Lecture 8

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For more details about the materials covered in this note, see Chapter 10.1 of Resnick [2] and Chapter A.4 of Durrett [1].

8.1 Radon-Nikodym Theorem

Definition 8.1. For two measures defined on the same measurable space $(\Omega, \mathcal{F})$, we say ν is absolutely continuous with respect to μ if $\mu(A) = 0$ implies $\nu(A) = 0$ for any $A \in \mathcal{F}$. This is often denoted by $\nu \ll \mu$. (Sometimes we also say μ dominates ν.)

We say μ, ν are equivalent and write $\mu \simeq \nu$ if $\mu \ll \nu$ and $\nu \ll \mu$. We say μ, ν are mutually singular, which is denoted by $\mu \perp \nu$, if there exist $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, $\mu(A^c) = \nu(B^c) = 0$.

Theorem 8.1 (Radon-Nikodym theorem)**.** *Let μ, ν be σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu \ll \mu$. Then there exists a Borel function $f \geq 0$ (measurable w.r.t. $\mathcal{F}$) such that, for any $A \in \mathcal{F}$,*

$$\nu(A) = \int_A f d\mu.$$

Further, f is unique μ-a.e. We call f the Radon-Nikodym derivative or the density of ν w.r.t. μ, and we write $f = d\nu/d\mu$, $d\nu = f d\mu$, $\nu(dx) = f(x)\mu(dx)$ or $d\nu(x) = f(x)d\mu(x)$.

Proof. See the textbook. □

Example 8.1. If μ is the Lebesgue measure, then the function f in Radon-Nikodym theorem is called the Lebesgue density. If the distribution (i.e. $\mathsf{P} \circ X^{-1}$) of a random variable X has a Lebesgue density, we say X is absolutely continuous.

Example 8.2. Consider $(\Omega, \mathcal{P}(\Omega), \mathsf{P})$ where $\Omega = \{\omega_1, \omega_2, \dots\}$ is a discrete set. Then the density function w.r.t. the counting measure is simply given by $f(\omega_i) = \mathsf{P}(\{\omega_i\})$, which is often called the probability mass function.

Example 8.3. The Cantor distribution is the uniform distribution on the Cantor set (which is a subset of $[0, 1]$). See Example 1.2.7 in Durrett [1]. The distribution function is continuous. However, the Lebesgue measure of the Cantor set is zero; that is, the Cantor distribution and the Lebesgue measure are singular. It has no density w.r.t. the counting measure either, since it has no point masses. We say it is a singular distribution.

8.2 Properties of Radon-Nikodym derivatives

Proposition 8.1. *Measures mentioned below are assumed to be σ-finite and defined on the measurable space $(\Omega, \mathcal{F})$.*

(i) If $\nu_1, \nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$ and

$$\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{\mu}, \qquad \mu - a.e.$$

($\nu_1 + \nu_2$ is defined by $(\nu_1 + \nu_2)(A) = \nu_1(A) + \nu_2(A)$ for any $A \in \mathcal{F}$.)

(ii) If $\nu \ll \mu$ and $f \geq 0$, then

$$\int f d\nu = \int f \left(\frac{d\nu}{d\mu} \right) d\mu.$$

(iii) If $\pi \ll \nu \ll \mu$, then

$$\frac{d\pi}{d\mu} = \frac{d\pi}{d\nu} \frac{d\nu}{d\mu}, \qquad \mu - a.e.$$

(iv) If $\nu \ll \mu$ and $\mu \ll \nu$,

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu} \right)^{-1}, \qquad \mu - a.e.$$

Proof of part (i). Details of the first two steps are omitted.

Step (1). Prove that $\nu_1 + \nu_2$ is a σ-finite measure.

Step (2). Prove that $\nu_1 + \nu_2 \ll \mu$.

Step (3). Consider any set $A \in \mathcal{F}$.

$$
\begin{aligned}
& (\nu_1 + \nu_2)(A) \\
={}& \nu_1(A) + \nu_2(A) && \text{(by definition of } \nu_1 + \nu_2\text{)} \\
={}& \int_A \frac{d\nu_1}{d\mu} d\mu + \int_A \frac{d\nu_2}{d\mu} d\mu && \text{(by the R-N theorem)} \\
={}& \int_A \left(\frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu} \right) d\mu && \text{(by linearity of Lebesgue integrals).}
\end{aligned}
$$

Finally, by the uniqueness part of the R-N theorem, $d\nu_1/d\mu + d\nu_2/d\mu$ must be equal to $d(\nu_1 + \nu_2)/d\mu$, μ-a.e. □

Proof of part (ii). Try it yourself. Recall how we construct the Lebesgue integral: start from indicator functions and simple functions, and then move on to consider more general choices of f. □

Proof of part (iii). The existence of $d\pi/d\nu$, $d\pi/d\mu$, and $d\nu/d\mu$ are guaranteed by the R-N theorem. To prove the claim, note that for any $A \in \mathcal{F}$,

$$
\begin{aligned}
\pi(A) &= \int_A \frac{d\pi}{d\nu} d\nu && \text{(by the R-N theorem)} \\
&= \int_A \frac{d\pi}{d\nu} \frac{d\nu}{d\mu} d\mu && \text{(by part (ii) and letting } f = d\pi/d\nu\text{)} .
\end{aligned}
$$

Apply the uniqueness part of the R-N theorem to conclude the proof. □

Proof of part (iv). The proof is similar to that of part (iii). □

Proposition 8.2. *Let μ_i, ν_i be σ-finite measures on $(\Omega_i, \mathcal{F}_i)$ for $i = 1, 2$. If $\nu_i \ll \mu_i$ for $i = 1, 2$, then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and*

$$
\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(\omega_1, \omega_2) = \frac{d\nu_1}{d\mu_1}(\omega_1) \cdot \frac{d\nu_2}{d\mu_2}(\omega_2), \qquad (\mu_1 \times \mu_2) - a.e.
$$

Sketch of the proof. First, use Fubini's theorem to show that for any measurable rectangle $A_1 \times A_2$,

$$
(\nu_1 \times \nu_2)(A_1 \times A_2) = \int_{A_1 \times A_2} \frac{d\nu_1}{d\mu_1}(\omega_1) \cdot \frac{d\nu_2}{d\mu_2}(\omega_2)(\mu_1 \times \mu_2) d(\omega_1, \omega_2).
$$

Then one can apply Dynkin's π-λ theorem. Alternatively, define another measure ν on the product space by letting

$$\nu(A) = \int_A \frac{d\nu_1}{d\mu_1}(\omega_1) \cdot \frac{d\nu_2}{d\mu_2}(\omega_2)(\mu_1 \times \mu_2)d(\omega_1, \omega_2),$$

for any $A \in \mathcal{F}_1 \times \mathcal{F}_2$. By Theorem 6.1, $\nu = \nu_1 \times \nu_2$, and the claim follows from the R-N theorem. □

References

[1] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge university press, 2019.

[2] Sidney Resnick. *A Probability Path*. Springer, 2019.