Lecture 21

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The materials covered in this note are from the unpublished book of Cox [1]. For a textbook reference, see Chapter 1.5 of Shao [2].

21.1 Basic asymptotic notations

Definition 21.1. Let $\{a_n\}_{n\geq 1}$ be a sequence of *positive* real numbers and $\{b_n\}_{n\geq 1}$ be a sequence of real numbers.

(i) We write $b_n = O(a_n)$ if $\limsup_{n\to\infty} |b_n|/a_n < \infty$.

(ii) We write $b_n = o(a_n)$ if $\lim_{n\to\infty} |b_n|/a_n = 0$.

(iii) For positive $\{b_n\}_{n\geq 1}$, we write $a_n \asymp b_n$ if $a_n = O(b_n)$ and $b_n = O(a_n)$.

(iv) For positive $\{b_n\}_{n\geq 1}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$.

Proposition 21.1. *Informally, we have the following arithmetic rules:*[1]

(i) $O(a_n) + O(a_n) = O(a_n)$*, and* $o(a_n) + o(a_n) = o(a_n)$.

(ii) $O(O(a_n)) = O(a_n)$*, and* $o(o(a_n)) = o(a_n)$.

(iii) $O(o(a_n)) = o(a_n)$*, and* $o(O(a_n)) = o(a_n)$.

(iv) $O(a_n)O(b_n) = O(a_nb_n)$*, and* $o(a_n)o(b_n) = o(a_nb_n)$.

(v) $O(a_n)o(b_n) = o(a_nb_n)$.

Proof. Since lim sup is subadditive, we have

$$\limsup_{n\to\infty} \frac{|b_n| + |c_n|}{a_n} \leq \limsup_{n\to\infty} \frac{|b_n|}{a_n} + \limsup_{n\to\infty} \frac{|c_n|}{a_n},$$

which immediately yields part (i). Next, for $a_n, b_n > 0$,

$$\limsup_{n\to\infty} \frac{|c_n|}{a_n} = \limsup_{n\to\infty} \frac{|c_n|}{b_n}\frac{b_n}{a_n} \leq \limsup_{n\to\infty} \frac{|c_n|}{b_n} \limsup_{n\to\infty} \frac{b_n}{a_n},$$

[1] For example, the rule "$O(a_n) + O(a_n) = O(a_n)$" actually means that if $b_n = O(a_n)$ and $c_n = O(a_n)$, then $b_n + c_n = O(a_n)$.

provided that both supremums on the right-hand side are finite. Both parts (ii) and (iii) can be easily verified using the above inequality. Parts (iv) and (v) can be shown by analogous arguments. □

Example 21.1. Suppose $X_1, X_2, \dots$ are i.i.d. with $E(X) = \mu, \mathrm{Var}(X) = \sigma^2, E(X^4) < \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function such that all derivatives up to order 4 exist and the fourth order derivative is bounded. Then,

$$E[f(\bar{X}_n)] = f(\mu) + \frac{\sigma^2 f''(\mu)}{2n} + O(n^{-2}),$$

which can be proven using the Taylor expansion. The notation $O(n^{-2})$ tells us the remainder term goes to zero at rate (not slower than) n^{-2}.

21.2 Probabilistic asymptotics

Definition 21.2. Let $\{X_n\}_{n \geq 1}$ be a sequence of random variables and $\{a_n\}_{n \geq 1}$ be a sequence of *positive* real numbers.

(i) We write $X_n = O_p(a_n)$ if for all $\delta > 0$, there exist $M_\delta, N_\delta < \infty$ such that for all $n \geq N_\delta$,

$$\mathsf{P}(|X_n|/a_n \leq M_\delta) \geq 1 - \delta.$$

(ii) We write $X_n = o_p(a_n)$ if $X_n/a_n \xrightarrow{P} 0$, i.e. for all $\delta > 0$, $\epsilon > 0$, there exist $N_{\delta,\epsilon} < \infty$ such that for all $n \geq N_{\delta,\epsilon}$,

$$\mathsf{P}(|X_n|/a_n \leq \epsilon) \geq 1 - \delta,$$

Proposition 21.2. *Informally , we have the following arithmetic rules:*

(i) $O_p(a_n) + O_p(a_n) = O_p(a_n)$, *and* $o_p(a_n) + o_p(a_n) = o_p(a_n)$.

(ii) $O_p(O(a_n)) = O_p(a_n)$, *and* $o_p(o(a_n)) = o_p(a_n)$.

(iii) $O_p(o(a_n)) = o_p(a_n)$, *and* $o_p(O(a_n)) = o(a_n)$.

(iv) $O_p(a_n)O_p(b_n) = O_p(a_n b_n)$, *and* $o_p(a_n)o_p(b_n) = o_p(a_n b_n)$.

(v) $O_p(a_n)o_p(b_n) = o_p(a_n b_n)$.

Proof. The proof is more complicated than the deterministic case, though in principle the two proofs are very similar. To see this, note that $b_n = O(a_n)$ is also equivalent to saying that there exist $N, M < \infty$ such that for all $n \geq N$, $|b_n|/a_n \leq M$. Here we only prove some of the rules.

First, consider the first statement of part (i). Let $X_n = O_p(a_n)$ and $Y_n = O_p(a_n)$ and fix $\delta > 0$. Then, by definition, there exist $M_1, M_2, N < \infty$ such that for all $n \geq N$,

$$\mathsf{P}(|X_n|/a_n \leq M_1) \geq 1 - \delta/2, \qquad \mathsf{P}(|Y_n|/a_n \leq M_2) \geq 1 - \delta/2,$$

By the union bound,

$$\mathsf{P}\{(|X_n| + |Y_n|)/a_n \leq M_1 + M_2\} \geq \mathsf{P}(|X_n|/a_n \leq M_1,\ |Y_n|/a_n \leq M_2) \geq 1 - \delta,$$

which proves $X_n + Y_n = O_p(a_n)$.

Next, consider the first statement of part (ii). Our goal is to prove if $b_n > 0$, $b_n = o(a_n)$ and $X_n = O_p(b_n)$, then $X_n = O_p(a_n)$. For any $\delta > 0$, by definition, there exist M_δ, N_δ, C such that for all $n \geq N_\delta$,

$$b_n/a_n \leq C, \qquad \mathsf{P}(|X_n|/b_n \leq M_\delta) \geq 1 - \delta,$$

which immediately gives, for $n \geq N_\delta$,

$$\mathsf{P}(|X_n|/a_n \leq CM_\delta) \geq 1 - \delta.$$

Since δ is arbitrary, we obtain $X_n = O_p(a_n)$.

Finally, consider part (v). Let $X_n = O_p(a_n)$ and $Y_n = o_p(b_n)$. For all $\delta > 0$, $\epsilon > 0$, there exist $M_\delta, N < \infty$ such that for all $n \geq N$,

$$\mathsf{P}(|X_n|/a_n \leq M_\delta) \geq 1 - \delta/2, \qquad \mathsf{P}(|Y_n|/b_n \leq \epsilon/M_\delta) \geq 1 - \delta/2.$$

Apply the union bound we get

$$\mathsf{P}\left(\frac{|X_n Y_n|}{a_n b_n} \leq \epsilon\right) \geq 1 - \delta,$$

i.e. $X_n Y_n / a_n b_n \xrightarrow{P} 0$. □

Example 21.2. If $X_n \xrightarrow{D} X$, then $X_n = O_p(1)$ (i.e. $\{F_n\}$ is tight where F_n denotes the distribution function of X_n). In particular, every random variable is $O_p(1)$. Sometimes we call an $O_p(1)$ term "stochastically bounded".

Example 21.3. If $E|X_n| = O(a_n)$, then $X_n = O_p(a_n)$. To prove this, first note that for sufficiently large n, we have $E|X_n| \leq Ca_n$ for some $C < \infty$. For any $\delta > 0$, letting $M_\delta = C/\delta$ and applying Markov inequality, we obtain

$$\mathsf{P}(|X_n|/a_n \geq M_\delta) \leq \frac{E|X_n|\delta}{Ca_n} \leq \delta.$$

Hence, $X_n = O_p(a_n)$.

References

[1] Dennis D. Cox. *The Theory of Statistics and Its Applications*. Unpublished.

[2] Jun Shao. *Mathematical statistics*. Springer Science & Business Media, 2003.