Lecture 19

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For more details about the materials covered in this note, see Chapters 9.2 to 9.6 of Resnick [2] and Chapter 3.3 of Durrett [1].

19.1 Properties of the function e^{ix}

We use i to denote the imaginary unit.

Theorem 19.1 (Euler's formula)**.** *For any* $x \in \mathbb{R}$*,* $e^{ix} = \cos x + i \sin x$*.*

Proof. One way to prove the formula is to use Taylor expansion. □

Theorem 19.2 (Taylor expansion of e^{ix})**.** *For any* $x \in \mathbb{R}$*,*

$$\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \leq \min \left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}.$$

Proof. See the textbook. □

Remark 19.1. Assume the moment generating function of $|X|$ is finite in a neighborhood of 0, i.e. for some $\delta > 0$,

$$E[e^{t|X|}] = \sum_{n=0}^{\infty} \frac{t^n E|X|^n}{n!} < \infty, \qquad \forall\, t \in (-\delta, \delta).$$

This implies $\lim_{n\to\infty} t^n E|X|^n / n! = 0$ for each $t \in (-\delta, \delta)$. By Theorem 19.2 and Jensen's inequality,

$$\left| E[e^{itX}] - \sum_{k=0}^{n} \frac{(it)^k}{k!} E[X^k] \right| \leq \frac{2t^n E|X|^n}{n!}.$$

The right-hand side converges to zero as $n \to \infty$. That is,

$$E[e^{itX}] = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E[X^k], \qquad \forall\, t \in (-\delta, \delta).$$

19.2 Basic properties of characteristic functions

Definition 19.1. The characteristic function of a random variable X is

$$\phi_X(t) = E[e^{itX}], \quad t \in \mathbb{R}.$$

Example 19.1. Let $X \sim N(\mu, \sigma^2)$. Using contour integral, we can compute

$$\phi_X(t) = \exp\left(i\mu t - \frac{\sigma^2}{2} t^2\right).$$

There are other ways to show this. For example, Resnick uses Taylor expansion and the MGF while Durrett uses an ordinary differential equation.

Proposition 19.1 (Properties of characteristic functions)**.** *For any $t \in \mathbb{R}$:*

(i) $\phi_X(t) = E[\cos(tX)] + i\,E[\sin(tX)]$*;*

(ii) $|\phi_X(t)| \le 1$ *and in particular* $\phi_X(0) = 1$*;*[1]

(iii) $\phi_X(-t) = \bar{\phi}_X(t)$ *where* $\bar{\phi}$ *denotes the complex conjugate.*

(iv) $\phi_X(t)$ *is uniformly continuous in* t*.*

Proof. Part (i) and (iii) follow from Theorem 19.1. For part (ii), note that $g(x, y) = \sqrt{x^2 + y^2}$ is a convex function. Thus, by Jensen's inequality,

$$|\phi_X(t)| \le E|e^{itX}| = 1.$$

For part (iv), by Jensen's inequality and the convexity of the modulus,

$$|\phi_X(t + h) - \phi_X(t)| \le E|e^{i(t+h)X} - e^{itX}| = E|e^{ihX} - 1|$$

where in the last step we have used the fact that $|z_1 z_2| = |z_1||z_2|$ for any complex numbers z_1, z_2. By the bounded convergence theorem, $E|e^{ihX} - 1| \to 0$ as $h \to 0$. Since this convergence does not depend on t, we obtain the uniform continuity of ϕ_X. □

Proposition 19.2. *Let $X_1, X_2, \dots$ be i.i.d. with characteristic function ϕ_X. Let $S_n = \sum_{i=1}^n (a_i X_i + b_i)$. Then, letting $c_n = \sum_{i=1}^n b_i$, we have*

$$\phi_{S_n}(t) = e^{itc_n} \prod_{i=1}^n \phi_X(a_i t).$$

[1] Here $|\cdot|$ denotes the modulus of a complex number: $|a + bi| = \sqrt{a^2 + b^2}$.

Proof. Try it yourself. □

Proposition 19.3. *If $E|X|^k < \infty$, then $\phi_X^{(k)}(0) = i^k E[X^k]$, where $\phi_X^{(k)}$ denotes the k-th derivative of ϕ_X.*

Proof. This can be proven by induction. Let's first show that if $E|X| < \infty$,

$$\phi'_X(t) = E[iXe^{itX}], \quad \forall t \in \mathbb{R}.$$

Consider the "error"

$$\begin{aligned}\frac{\phi_X(t+h) - \phi_X(t)}{h} - E[iXe^{itX}] &= E\left(\frac{e^{i(t+h)X} - e^{itX} - ihXe^{itX}}{h}\right) \\ &= E\left(e^{itX}\frac{e^{ihX} - 1 - ihX}{h}\right).\end{aligned}$$

It suffices to show that the above expression goes to zero as $h \downarrow 0$. By Theorem 19.2, we have (both bounds on the right-hand side are useful!)

$$\left|\frac{e^{ihX} - 1 - ihX}{h}\right| \leq \min\left\{2|X|, \frac{h|X|^2}{2}\right\}.$$

Since $E|X| < \infty$ and $|e^{itX}| = 1$, by DCT,

$$\lim_{h\downarrow 0} E\left(e^{itX}\frac{e^{ihX} - 1 - ihX}{h}\right) = E\left(e^{itX}\lim_{h\downarrow 0}\frac{e^{ihX} - 1 - ihX}{h}\right) = 0.$$

Now let's assume that

$$\text{if } E|X|^k < \infty, \quad \text{then } \phi_X^{(k)}(t) = E[(iX)^k e^{itX}].$$

Consider

$$\frac{\phi_X^{(k)}(t+h) - \phi_X^{(k)}(t)}{h} - E[(iX)^{k+1}e^{itX}] = E\left(e^{itX}(iX)^k\frac{e^{ihX} - 1 - ihX}{h}\right).$$

Apply the same argument to obtain that

$$\left|(iX)^k\frac{e^{ihX} - 1 - ihX}{h}\right| \leq \min\left\{2|X|^{k+1}, \frac{h|X|^{k+2}}{2}\right\}.$$

Hence, for every fixed $X = x$, the left-hand side goes to zero as $h \downarrow 0$. If $E|X|^{k+1} < \infty$, one can apply DCT to conclude that

$$\phi_X^{(k+1)} = \lim_{h\downarrow 0}\frac{\phi_X^{(k)}(t+h) - \phi_X^{(k)}(t)}{h} = E[(iX)^{k+1}e^{itX}].$$

Letting $t = 0$, we obtain the asserted formula. □

Example 19.2. Let X be a continuous random variable with density $f(x) = \mathbb{1}_{\{|x|>2\}} c/(x^2 \log |x|)$ where c is some normalization constant. The expectation does not exist since $\int_2^\infty 1/(x \log x) dx = \infty$. But it can be proven that $\phi'_X(0)$ exists and is equal to zero.

19.3 Uniqueness and continuity of characteristic functions

Theorem 19.3 (Inversion formula)**.** *Let $\phi(t) = \int e^{itx}\mu(dx)$ be the characteristic function for some distribution μ. For any $a < b$,*

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t) dt = \mu((a,b)) + \frac{1}{2}\mu(\{a,b\}).$$

Proof. Define

$$I_T = \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t) dt = \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \left(\int e^{itx}\mu(dx) \right) dt.$$

Note that $(e^{-ita} - e^{-itb})/(it) = \int_a^b e^{-ity} dy$ and thus

$$\left| \frac{e^{-ita} - e^{-itb}}{it} \phi(t) \right| = \left| \frac{e^{-ita} - e^{-itb}}{it} \right| = \left| \int_a^b e^{-ity} dy \right| \leq \int_a^b |e^{-ity}| dy = b - a.$$

Hence, we can apply Fubini's theorem to get

$$I_T = \int_{\mathbb{R}} \left\{ \int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it} dt \right\} \mu(dx).$$

Applying Euler's formula and noting that cos is an even function, we obtain

$$\begin{aligned} I_T &= \int_{\mathbb{R}} \left\{ \int_{-T}^{T} \frac{\sin[t(x-a)] - \sin[t(x-b)]}{t} dt \right\} \mu(dx) \\ &= \int_{\mathbb{R}} \left\{ R(x-a, T) - R(x-b, T) \right\} \mu(dx), \end{aligned}$$

where we let

$$R(\theta, T) = \int_{-T}^{T} t^{-1} \sin(\theta t) dt = \int_{-\theta T}^{\theta T} u^{-1} \sin u \, du.$$

Define $h(u) = u^{-1} \sin u$. Note that $|h(u)| \leq 1$, for any $u \in \mathbb{R}$. Further, by Lemma 19.1 below, $\lim_{M\to\infty} \int_{-M}^{M} h(u)du = \pi$. Therefore, $\sup_\theta R(\theta, T) = C < \infty$ for some constant C. By bounded convergence theorem,

$$\begin{aligned}\lim_{T\to\infty} I_T &= \int_{\mathbb{R}} \lim_{T\to\infty} [R(x-a,T) - R(x-b,T)]\mu(dx) \\ &= 2\pi\mu((a,b)) + \pi\mu(\{a,b\})\end{aligned}$$

where we have used

$$\lim_{T\to\infty} R(\theta, T) = \begin{cases} \pi & \theta > 0 \\ 0 & \theta = 0 \\ -\pi & \theta < 0, \end{cases}$$

which again follows from Lemma 19.1. □

Lemma 19.1. *The improper Riemann integral* $\int_{-\infty}^{\infty} u^{-1} \sin u du = \pi$.

Proof. We omit the proof here. This integral is known as Dirichlet integral. The corresponding Lebesgue integral is not defined. □

Corollary 19.1. $\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\phi(t)dt$.

Proof. Try it yourself. □

Corollary 19.2. *If* $\int |\phi(t)|dt < \infty$, *then* μ *has a bounded and continuous density function given by*

$$f(x) = \frac{1}{2\pi} \int e^{-itx}\phi(t)dt.$$

Proof. In the proof of Theorem 19.3, we have shown that for $b > a$,

$$\left| \frac{e^{-ita} - e^{-itb}}{it} \phi(t) \right| \leq (b-a)|\phi(t)|.$$

Since $\int |\phi(t)|dt < \infty$, by the inversion formula,

$$\mu((a,b)) + \frac{1}{2}\mu(\{a,b\}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \phi(t)dt \leq \frac{b-a}{2\pi} \int_{-\infty}^{\infty} |\phi(t)|dt < \infty.$$

Letting $b \downarrow a$, we get $\mu(\{a\}) \leq \mu(\{a, b\}) \leq 0$ and thus $\mu \ll m$ (m denotes the Lebesgue measure.) Applying Fubini's theorem, we find that

$$\begin{aligned}\mu((a,b)) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_a^b e^{-ity}dy\phi(t)dt \\ &= \int_a^b \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\phi(t)dt\right\}dy.\end{aligned}$$

Hence, $f(y) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-ity}\phi(t)dt$ is the Radon-Nikodym derivative,[2] which is essentially unique (i.e. unique up to a Lebesgue-null set). By the assumption $\int \phi(t)dt < \infty$, f is bounded; further, it is continuous due to DCT. □

Example 19.3. Let X be a Cauchy random variable with density $f(x) = \pi^{-1}(1+x^2)^{-1}$. Direct calculation of its characteristic function seems difficult. However, for another random variable Y with density $f(y) = e^{-|y|}/2$, it is easy to compute $\phi_Y(t) = (1+t^2)^{-1}$. By the inversion formula, this implies

$$\frac{1}{2}e^{-|y|} = \frac{1}{2\pi}\int \frac{1}{1+t^2}e^{-ity}dt.$$

Hence,

$$\phi_X(t) = \int \frac{e^{-itx}}{\pi(1+x^2)}dx = e^{-|t|}.$$

Theorem 19.4 (Uniqueness theorem)**.** *The characteristic function uniquely determines the probability distribution.*

Proof. By the inversion formula, the characteristic function uniquely determines the value of $\mu((a,b))$ for any $a < b$. By Dynkin's π-λ theorem, this means $\mu(B)$ is uniquely defined for any Borel set B. □

Example 19.4. Consider a continuous random variable X with density

$$f(x) = \frac{1-\cos x}{\pi x^2}, \quad x \in \mathbb{R}.$$

Consider another discrete random variable Y with

$$\mathsf{P}(Y=0) = \frac{1}{2}, \quad \mathsf{P}(Y=(2k-1)\pi) = \frac{2}{(2k-1)^2\pi^2}, \ k = 0, \pm 1, \pm 2, \ldots.$$

[2] One can invoke π-λ theorem to show that $\mu(B) = \int_B f(y)dy$ for any Borel set B.

Then, the characteristic functions of X and Y are

$$\begin{aligned}\phi_X(t) &= (1-|t|)\,\mathbb{1}_{\{|t|\leq 1\}},\\ \phi_Y(t) &= \frac{1}{2}+\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\{(2k-1)\pi t\}}{(2k-1)^2}.\end{aligned}$$

Surprisingly, for any $t \in [-1, 1]$, we have $\phi_X(t) = \phi_Y(t)$.

Theorem 19.5 (Continuity theorem)**.** *Let $\{\mu_n\}_{n\geq 1}$ be probability distributions with characteristic function $\{\phi_n\}_{n\geq 1}$.*

(i) If μ_n converges weakly to some distribution μ, then $\phi_n(t) \to \phi(t)$ for every t where ϕ is the characteristic function of μ.

(ii) If $\phi_n(t) \to \phi(t)$ for every t and ϕ is a function continuous at 0, then μ_n converges weakly to some distribution μ with characteristic function ϕ.

Proof. Since e^{itx} is a bounded and continuous function for every t, by Theorem 18.2, we immediately obtain part (i). The proof of part (ii) consists of two steps.

Step 1. We prove $\{\mu_n\}$ is tight, i.e. for any $\epsilon > 0$, there exist $C, N < \infty$ such that $\mu_n(\{x : |x| > C\}) \leq \epsilon$ for every $n \geq N$. Let $\epsilon > 0$ be fixed. Since $\phi(0)$ is always 1 and ϕ is assumed to be continuous at zero, there exists δ such that $|1 - \phi(t)| < \epsilon/2$ for any $t \in (-\delta, \delta)$. Since $\phi_n(t) \to \phi(t)$, by BCT, there exists N such that for any $n \geq N$,

$$\left|\frac{1}{\delta}\int_{-\delta}^{\delta}(1-\phi_n(t))dt - \frac{1}{\delta}\int_{-\delta}^{\delta}(1-\phi(t))dt\right| < \epsilon,$$

which further yields that

$$\frac{1}{\delta}\int_{-\delta}^{\delta}(1-\phi_n(t))dt < 2\epsilon.$$

On the other hand,

$$
\begin{aligned}
\frac{1}{\delta}\int_{-\delta}^{\delta}(1-\phi_n(t))dt &= 2 - \frac{1}{\delta}\int_{-\delta}^{\delta}\left\{\int e^{itx}\mu_n(dx)\right\}dt \\
&= 2 - \int_{\mathbb{R}}\left\{\int_{-\delta}^{\delta}\frac{e^{itx}}{\delta}dt\right\}\mu_n(dx) && \text{(by Fubini's theorem)} \\
&= 2 - \int_{\mathbb{R}}\frac{2\sin(\delta x)}{\delta x}\mu_n(dx) && \text{(by Euler's formula)} \\
&= 2\int_{\mathbb{R}}\left\{1-\frac{\sin(\delta x)}{\delta x}\right\}\mu_n(dx) \\
&\geq 2\int_{|x|>2/\delta}\left\{1-\frac{\sin(\delta x)}{\delta x}\right\}\mu_n(dx) \\
&\geq 2\int_{|x|>2/\delta}\left\{1-\frac{|\sin(\delta x)|}{|\delta x|}\right\}\mu_n(dx) \\
&\geq \int_{|x|>2/\delta}\mu_n(dx) = \mu_n(\{x : |x| > 2/\delta\}).
\end{aligned}
$$

By choosing $C = 2/\delta$, we conclude that $\{\mu_n\}$ is tight.

Step 2. By Helly's selection theorem and the tightness of $\{\mu_n\}$, there exists a subsequence $\{\mu_{n_k}\}_{k\geq 1}$ such that μ_{n_k} converges weakly to some distribution μ. By part (i), the characteristic function of μ is ϕ.

Now we prove that μ_n converges to μ by contradiction. Let F_n and F be the distribution functions of μ_n and μ, respectively. If we don't have the asserted weak convergence, there exists some point x, which is a continuous point of F, such that $F_n(x)$ does not converge to $F(x)$; that is, there exists some $\eta > 0$ and a subsequence $\{F_{m_k}\}_{k\geq 1}$ such that $|F_{m_k}(x) - F(x)| \geq \eta$. But by Helly's selection theorem and the tightness of $\{F_n\}$, there exists a subsequence of $\{F_{m_k}\}_{k\geq 1}$, say $\{F_{m_{k(j)}}\}_{j\geq 1}$, that converges to a proper distribution function, say F'. Let ϕ' be its characteristic function. However, the convergence of ϕ_n implies that $\phi'(t) = \phi(t)$ for every t, and by the uniqueness theorem we have $F' = F$, which gives the contradiction. □

Example 19.5. Let μ_n be a normal distribution with mean zero and variance n, and thus $\phi_n(t) = e^{-nt^2/2}$. Clearly, for every $t \neq 0$, $\phi_n(t) \to 0$, i.e. the limit ϕ is not continuous at zero. The sequence $\{\mu_n\}$ does not converge weakly since $\mu_n((-\infty, x]) \to 1/2$ for every $x \in \mathbb{R}$.

References

[1] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge university press, 2019.

[2] Sidney Resnick. *A Probability Path*. Springer, 2019.

[3] Jordan M Stoyanov. *Counterexamples in probability*. Courier Corporation, 2013.