Lecture 15

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For more details about the materials covered in this note, see Chapter 7.2 of Resnick [2] and Chapter 2.2 of Durrett [1].

15.1 Weak law of large numbers for triangular arrays

Theorem 15.1 (WLLN for triangular arrays)**.** *Consider random variables* $\{X_{n,k} : 1 \le k \le n,\ n \ge 1\}$*, which is often called a triangular array. For each* n*, assume that* $X_{n,1}, \dots, X_{n,n}$ *are independent. Let* $b_n > 0$ *with* $b_n \to \infty$ *and let* $Y_{n,k} = X_{n,k} \mathbb{1}_{\{|X_{n,k}| \le b_n\}}$ *(truncation). Suppose that as* $n \to \infty$*,*

(i) $\sum_{k=1}^{n} \mathsf{P}(|X_{n,k}| > b_n) \to 0$*;*

(ii) $\frac{1}{b_n^2} \sum_{k=1}^{n} E(Y_{n,k}^2) \to 0$*.*

Finally, let $S_n = X_{n,1} + \dots + X_{n,n}$ *and* $a_n = \sum_{k=1}^{n} E(Y_{n,k})$*, then*

$$\frac{S_n - a_n}{b_n} \xrightarrow{P} 0.$$

Proof. Let $T_n = Y_{n,1} + \dots + Y_{n,n}$. Notice that

$$\mathsf{P}\left(\left| \frac{S_n - a_n}{b_n} \right| > \epsilon \right) \le \mathsf{P}(S_n \ne T_n) + \mathsf{P}\left(\left| \frac{T_n - a_n}{b_n} \right| > \epsilon \right),$$

by the union bound. Now we analyze the two terms on the r.h.s. separately. For the first term, note that if $S_n \ne T_n$, there is at least one k such that $Y_{n,k} \ne X_{n,k}$. Thus, by union bound again,

$$\mathsf{P}(S_n \ne T_n) \le \mathsf{P}\left(\cup_{k=1}^{n} \{Y_{n,k} \ne X_{n,k}\} \right) \le \sum_{k=1}^{n} \mathsf{P}(|X_{n,k}| > b_n) \to 0,$$

by assumption (i). For the second term, apply Markov's inequality and the inequality $\text{Var}(X) \le EX^2$ to obtain that

$$\begin{aligned}\mathsf{P}\left(\left|\frac{T_n - a_n}{b_n}\right| > \epsilon\right) &\le \frac{1}{\epsilon^2} E\left|\frac{T_n - a_n}{b_n}\right|^2 = \frac{1}{\epsilon^2}\text{Var}\left(\frac{T_n}{b_n}\right) \\ &= \frac{\text{Var}(T_n)}{\epsilon^2 b_n^2} = \frac{1}{\epsilon^2 b_n^2}\sum_{k=1}^n \text{Var}(Y_{n,k}) \\ &\le \frac{1}{\epsilon^2 b_n^2}\sum_{k=1}^n E|Y_{n,k}|^2 \to 0,\end{aligned}$$

where the last step follows from assumption (ii). Since $\epsilon > 0$ is arbitrary, we get the asserted convergence in probability. □

Example 15.1 (St. Petersburg paradox)**.** A casino offers the following game: you keep flipping a (fair) coin until you get a tail and, the payout is 2^k dollars where k is the total number of flips. For example, if the first flip is a head and the second is a tail, then you get 4 dollars. Let X be the payout. Clearly, $\mathsf{P}(X = 2^k) = 2^{-k}$ and thus $E[X] = \infty$. What would be a fair price to play this game? One possible solution is to use WLLN. Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with the same distribution as X. One can apply WLLN for triangular arrays with $X_{n,k} = X_k$,

$$a_n = n \log_2 n + n \log_2(\log_2 n), \quad b_n = n \log_2 n$$

to show that $S_n/(n \log_2 n) \xrightarrow{P} 1$. (See Durrett's book for details.) So if you plan to play the game $1,000$ times, on average you will win $\log_2 1000 \approx 10$ dollars each time and thus 10 dollars is arguably a fair price.

15.2 Special cases of WLLN

Theorem 15.2 (Feller's WLLN)**.** *For an i.i.d. sequence of random variables* $\{X_n\}_{n\ge1}$ *with* $\lim_{x\to\infty} x\mathsf{P}(|X_1| > x) = 0$*, we have*

$$\frac{S_n}{n} - E(X_1 \mathbb{1}_{\{|X_1|\le n\}}) \xrightarrow{P} 0.$$

Proof. We apply the WLLN for triangular arrays with $b_n = n$ and $X_{n,k} = X_k$. By the i.i.d. assumption, condition (i) is automatically satisfied. Further,

$a_n/b_n = E[X_1 \mathbb{1}_{|X_1|\le n}]$. So we only need to verify condition (ii). For a non-negative random variable Z and $p > 0$, we have $E(Z^p) = \int_0^\infty p z^{p-1} \mathsf{P}(Z > z)dz$. Thus, by letting $Y_{n,1} = X_1 \mathbb{1}_{\{|X_1|\le n\}}$,

$$\begin{aligned} E[Y_{n,1}^2] &= \int_0^\infty 2y\mathsf{P}(|X_1|\mathbb{1}_{\{|X_1|\le n\}} > y)dy \\ &= \int_0^n 2y\mathsf{P}(|X_1|\mathbb{1}_{\{|X_1|\le n\}} > y)dy \\ &\le \int_0^n 2y\mathsf{P}(|X_1| > y)dy. \end{aligned}$$

But by the assumption that $y\mathsf{P}(|X_1| > y) \to 0$, we have

$$\frac{1}{n}\int_0^n 2y\mathsf{P}(|X_1| > y)dy \to 0, \qquad \text{as } n \to \infty, \tag{1}$$

which shows that condition (ii) in Theorem 15.1 is satisfied. Intuitively, (1) is true because the l.h.s. can be interpreted as the average of $2y\mathsf{P}(|X_1| > y)$ which goes to zero (you may recall Cesaro mean.) We present the complete proof in the following lemma. □

Lemma 15.1. *Let* $g\colon [0,\infty) \to [0,\infty)$ *be a function such that* $\lim_{x\to\infty} g(x) = 0$ *and* $\sup_{0\le x\le n} g(x) < \infty$ *for every* n. *Then,* $n^{-1}\int_0^n g(x)dx \to 0$ *as* $n \to \infty$.

Proof. For any $\epsilon > 0$, there exists $K = K(\epsilon) < \infty$ such that $g(x) \le \epsilon$ for all $x \ge K$. Further, $\sup_{0\le x\le K} g(x) = M(K) < \infty$ by the assumption. Then,

$$\begin{aligned} \int_0^n g(x)dx &= \int_0^K g(x)dx + \int_K^n g(x)dx \\ &\le KM + (n-K)\epsilon. \end{aligned}$$

Hence, $n^{-1}\int_0^n g(x)dx < n^{-1}KM + \epsilon$. Note that both K and M only depend on ϵ. We can taking lim sup on both sides and get

$$\limsup_{n\to\infty} \frac{1}{n}\int_0^n g(x)dx < \epsilon.$$

Since ϵ is arbitrary and $g(x) \ge 0$, this means that $n^{-1}\int_0^n g(x)dx \to 0$. □

Example 15.2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that each follows the Cauchy distribution, i.e.

$$\mathsf{P}(X_i \leq x) = \int_{-\infty}^{x} \frac{dt}{\pi(1+t^2)}.$$

As $x \to \infty$, we have

$$\mathsf{P}(|X_i| > x) = 2\int_{x}^{\infty} \frac{dt}{\pi(1+t^2)} \sim \frac{2}{\pi}\int_{x}^{\infty} \frac{dt}{t^2} = \frac{2}{\pi x}.$$

The assumption of Feller's WLLN does not hold, and in fact S_n/n does not converge in probability.

Example 15.3. Let $\{X_n\}$ be i.i.d. and symmetric random variables with distribution function

$$1 - F(x) = \frac{e}{2x\log x}, \qquad \text{for } x \geq e.$$

(This implies $\mathsf{P}(X \in (-e, e)) = 0$.) One can check that $E[X^+] = E[X^-] = \infty$ and thus the expectation does not exist. However, $\lim_{x\to\infty} n\mathsf{P}(|X_1| > n) = e/\log n \to 0$, and thus the assumption of Feller's WLLN is satisfied. Further, $E[X_1 \mathbb{1}_{\{|X_1| \leq n\}}] = 0$ for every n by symmetry. Hence, $S_n/n \xrightarrow{P} 0$.

Theorem 15.3 (Khintchin's WLLN)**.** *For an i.i.d. sequence* $\{X_n\}_{n\geq 1}$ *with mean* μ *and* $E|X_1| < \infty$, *we have* $S_n/n \xrightarrow{P} \mu$.

Proof. This is a special case of Feller's WLLN. To prove this, note that $E|X_1| < \infty$ implies that

$$n\mathsf{P}(|X_1| > n) = E[n\mathbb{1}_{\{|X_1|>n\}}] \leq E[|X_1|\mathbb{1}_{\{|X_1|>n\}}] \to 0,$$

by DCT. It also follows from DCT that $E(X_1\mathbb{1}_{\{|X_1|\leq n\}}) \to E[X_1]$. □

Theorem 15.4 (WLLN with finite variances)**.** *For an i.i.d. sequence* $\{X_n\}_{n\geq 1}$ *with mean* μ *and variance* $\sigma^2 < \infty$, *we have* $S_n/n \xrightarrow{P} \mu$.

Proof. This is just a special case of Khintchin's WLLN since finite variance implies that $E|X_1| < \infty$. □

References

[1] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge university press, 2019.

[2] Sidney Resnick. *A Probability Path*. Springer, 2019.