Lecture 14

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For more details about the materials covered in this note, see Chapters 6.3, 6.5 and 6.6 of Resnick [2] and Chapter 4.6 of Durrett [1].

14.1 More about almost sure convergence and convergence in probability

All random variables mentioned below, i.e. $X, \{X_n\}, Y, \{Y_n\}$, are assumed to be defined on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

Proposition 14.1. *$X_n \stackrel{P}{\to} X$ if and only if every subsequence $\{X_{n_k}\}$ contains a further subsequence $\{X_{n_{k(i)}}\}$ which converges almost surely to X.*

Proof. See the textbook. □

Remark 14.1. Suppose $X_n \stackrel{P}{\to} X$ and $X_n \stackrel{a.s.}{\to} X'$. By Proposition 14.1, there is a subsequence X_{n_k} such that $X_{n_k} \stackrel{a.s.}{\to} X$; that is, $\mathsf{P}(\{\omega : \lim_{k\to\infty} X_{n_k}(\omega) = X(\omega)\}) = 1$. But by Proposition 0.5 (subsequence of a convergent real sequence converges to the same limit), we also have $\mathsf{P}(\{\omega : \lim_{k\to\infty} X_{n_k}(\omega) = X'(\omega)\}) = 1$. Hence, by the union bound, $X(\omega) = X'(\omega)$, a.s.

Proposition 14.2. *Some results from the continuous mapping theorem.*

(i) If $X_n \stackrel{a.s.}{\to} X$, then $f(X_n) \stackrel{a.s.}{\to} f(X)$ for any continuous function f.

(ii) If $X_n \stackrel{P}{\to} X$, then $f(X_n) \stackrel{P}{\to} f(X)$ for any continuous function f.

(iii) If $X_n \stackrel{a.s.}{\to} X$ and $Y_n \stackrel{a.s.}{\to} Y$, then $aX_n + bY_n \stackrel{a.s.}{\to} aX + bY$ for $a, b \in \mathbb{R}$ and $X_nY_n \stackrel{a.s.}{\to} XY$.

(iv) If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$, then $aX_n + bY_n \stackrel{P}{\to} aX + bY$ for $a, b \in \mathbb{R}$ and $X_nY_n \stackrel{P}{\to} XY$.

Proof of part (i). If $X_n(\omega) \to X(\omega)$, then by continuity we have $f(X_n(\omega)) \to f(X(\omega))$. Since $X_n \to X$ a.s., we have $f(X_n) \stackrel{a.s.}{\to} f(X)$. □

Proof of part (ii). Apply Proposition 14.1 and Proposition 14.2 (i). □

Proof of the remaining part(s). Try it yourself. □

Theorem 14.1 (Dominated convergence theorem)**.** *If $X_n \xrightarrow{P} X$ and there exists an integrable random variable Z such that $|X_n| \leq Z$ for all n, then $E(X_n) \to E(X)$.*

Proof. The proof relies on a fact from analysis: if each subsequence of $(a_n)_{n\geq 1}$ contains a further subsequence that converges to a, then a_n converges to a. (You can prove it by contradiction.) Now you can prove Theorem 14.1 using Proposition 14.1. □

14.2 Uniform integrability

Definition 14.1. Let $\{X_t : t \in T\}$ be a family of integrable random variables (i.e. $E|X_t| < \infty$). We say this family is uniformly integrable if as $M \to \infty$, we have

$$\sup_{t\in T} E\left(|X_t| \mathbb{1}_{\{|X_t|>M\}}\right) = \sup_{t\in T} \int_{\{|X_t|>M\}} |X_t| d\mathsf{P} \to 0.$$

Proposition 14.3. *If there exists a random variable Z such that $E|Z| < \infty$ and $|X_t| \leq Z$ for every t, then $\{X_t\}$ is uniformly integrable.*

Proof. For every t, we have

$$\int_{\{|X_t|>M\}} |X_t| d\mathsf{P} \leq \int_{\{|X_t|>M\}} |Z| d\mathsf{P} \leq \int_{\{|Z|>M\}} |Z| d\mathsf{P}.$$

Hence, we only need to prove that if $E|Z| < \infty$, then $\int_{\{|Z|>M\}} |Z| d\mathsf{P} \to 0$ as $M \to \infty$. To show this, let $Y_M(\omega) = |Z(\omega)| \mathbb{1}_{\{|Z(\omega)|>M\}}$, which is dominated by $|Z|$. Since $E|Z| < \infty$, by the dominated convergence theorem, we have

$$\int_{\{|Z|>M\}} |Z| d\mathsf{P} = \int Y_M \, d\mathsf{P} \to \int \lim_{M\to\infty} Y_M \, d\mathsf{P} = \int 0 \, d\mathsf{P} = 0,$$

which concludes the proof. □

Remark 14.2. Proposition 14.3 implies that if $E(\sup_{t\in T} |X_t|) < \infty$, then $\{X_t\}_{t\in T}$ is uniformly integrable. If $\{X_t\}_{t\in T}$ is uniformly integrable, then there exists $M < \infty$ such that $\sup_t \int_{\{|X_t|>M\}} |X_t| d\mathsf{P} \leq \epsilon$ for some constant $\epsilon > 0$, which implies that

$$\sup_t E|X_t| \leq \sup_t \int (\mathbb{1}_{\{|X_t|>M\}} |X_t| + M) d\mathsf{P} \leq M + \epsilon < \infty.$$

Corollary 14.1. *If $E|X_t| < \infty$ for each t and the index set T is finite, then $\{X_t\}$ is uniformly integrable.*

Proof. Try it yourself. □

Proposition 14.4. *Consider two families of integrable random variable $\{X_t : t \in T\}$ and $\{Y_t : t \in T\}$. If $|X_t| \le |Y_t|$ for each t and $\{Y_t\}$ is uniformly integrable, then $\{X_t\}$ is uniformly integrable.*

Proof. Try it yourself. □

Theorem 14.2 (Crystal ball condition)**.** *For any $p > 0$, the family $\{|X_t|^p\}$ is uniformly integrable if $\sup_n E|X_n|^{p+\delta} < \infty$ for some $\delta > 0$.*

Proof. Try it yourself. □

Theorem 14.3. *A family of integrable random variables $\{X_t : t \in T\}$ is uniformly integrable if and only if the following two conditions are satisfied: (i) for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that for any $A \in \mathcal{F}$ with $\mathsf{P}(A) < \delta$, we have $\sup_{t\in T} \int_A |X_t|\, d\mathsf{P} < \epsilon$; (ii) $\sup_{t\in T} E|X_t| < \infty$.*

Proof. See the textbook. □

Example 14.1. Let $\{X_n\}$ be a sequence of random variables with $\mathsf{P}(X_n = n) = 1/n$ and $\mathsf{P}(X_n = 0) = 1 - 1/n$. Clearly, $\sup E|X_n| = 1$. However, this family is not uniformly integrable because

$$\int_{\{X_n > M\}} X_n \, d\mathsf{P} = \begin{cases} 1 & \text{if } M \le n, \\ 0 & \text{if } M > n, \end{cases}$$

which yields $\sup_{n\ge 1} \int_{\{X_n>M\}} X_n \, d\mathsf{P} = 1$ for every M.

14.3 More about convergence in L^p

Theorem 14.4 (Scheffe's lemma)**.** $X_n \xrightarrow{L^1} X$ *if and only if*

$$\sup_{A\in\mathcal{F}} \left| \int_A (X_n - X)\, d\mathsf{P} \right| \to 0.$$

Proof. First, assume $X_n \xrightarrow{L^1} X$. Then, without loss of generality, we may assume $X_n - X$ is integrable. Hence,

$$\begin{aligned}\sup_{A\in\mathcal{F}} \left| \int_A (X_n - X)\, d\mathsf{P} \right| &\le \sup_{A\in\mathcal{F}} \int_A |X_n - X|\, d\mathsf{P} \\ &\le \int_\Omega |X_n - X|\, d\mathsf{P} \\ &= E|X_n - X| \to 0.\end{aligned}$$

To prove the converse, note that

$$\begin{aligned}E|X_n - X| &= \int_{\{X_n > X\}} (X_n - X)\, d\mathsf{P} + \int_{\{X_n \le X\}} (X - X_n)\, d\mathsf{P} \\ &\le \left(\sup_A \int_A (X_n - X)\, d\mathsf{P}\right) + \left(\sup_A \int_A (X - X_n)\, d\mathsf{P}\right) \\ &\le 2 \sup_A \left| \int_A (X_n - X)\, d\mathsf{P} \right|.\end{aligned}$$

Hence, if the right-hand side converges to 0, $X_n \xrightarrow{L^1} X$. □

Corollary 14.2. *If $X_n \xrightarrow{L^1} X$ and $E|X| < \infty$, then $E(X_n) \to E(X)$.*

Proof. Try it yourself. □

Proposition 14.5. *Suppose $p \in [1, \infty)$. If $X_n \xrightarrow{L^p} X$ and $E|X_n|^p < \infty$ for every n, then $E|X_n|^p \to E|X|^p$ and $E|X|^p < \infty$.*

Proof. Since $X_n \xrightarrow{L^p} X$, there exists some k such that $\|X - X_k\|_p \le 1$. By the triangle inequality, this implies $E|X|^p < \infty$.

By the reverse triangle inequality, $|\|X_n\|_p - \|X\|_p| \le \|X_n - X\|_p$ for each n; that is,

$$\left| (E|X_n|^p)^{1/p} - (E|X|^p)^{1/p} \right| \le (E|X_n - X|^p)^{1/p} \to 0,$$

which implies $E|X_n|^p \to E|X|^p$. □

Theorem 14.5. *Suppose $p \in [1, \infty)$ and $\{X_n\}$ is a sequence of random variables such that $E|X_n|^p < \infty$ for all n. Then $X_n \xrightarrow{L^p} X$ if and only if (i) $\{|X_n|^p\}$ is uniformly integrable and (ii) $X_n \xrightarrow{P} X$.*

Proof. We divide the proof into two parts.

The "only if" part. Recall that the L^p convergence implies $X_n \xrightarrow{P} X$. Hence, we only need to show that $\{X_n\}$ is uniformly integrable. For any $M > 0$, define a continuous function ψ_M by

$$\psi_M(x) = \begin{cases} x & \text{if } x \in [0, M-1], \\ \text{linear} & \text{if } x \in [M-1, M], \\ 0 & \text{if } x \in [M, \infty). \end{cases}$$

Notice that $|x|^p - \psi_M(|x|)^p \geq |x|^p \mathbb{1}_{\{|x|>M\}}$ for any x and thus

$$\int_{\{|X_n|>M\}} |X_n|^p \, d\mathsf{P} \leq \int \{|X_n|^p - \psi_M(|X_n|)^p\} \, d\mathsf{P} = E|X_n|^p - E[\psi_M(|X_n|)^p]$$
$$\leq \{E|X_n|^p - E|X|^p\} + \{E|X|^p - E[\psi_M(|X|)^p]\} + \{E[\psi_M(|X|)^p] - E[\psi_M(|X_n|)^p]\}.$$

We now bound the three terms in the last line separately.

(i) Observe that $\psi_M(y) \to y$ pointwise as $M \to \infty$. Further, $\psi_M(y) \leq y$. Applying the dominated convergence theorem, we get $E|\psi_M(|X|)^p| \to E|X|^p$ (note that by Proposition 14.5, $E|X|^p < \infty$.) Hence, for any $\epsilon > 0$, we can find a sufficiently large M_1 such that

$$E|X|^p - E|\psi_{M_1}(|X|)^p| \leq \epsilon/3.$$

(ii) Let M be fixed and $n \to \infty$, then $\psi_M(|X_n|)^p \xrightarrow{P} \psi_M(|X|)^p$ since ψ_M is continuous. Since $\psi_M(|X_n|)^p \leq (M-1)^p$, by Theorem 14.1, we have $E[\psi_M(|X_n|)^p] \to E[\psi_M(|X|)^p]$ as $n \to \infty$. Therefore, we can also find a sufficiently large $N_1 = N_1(M)$ such that for all $n \geq N_1$,

$$E[\psi_M(|X|)^p] - E[\psi_M(|X_n|)^p] \leq \epsilon/3.$$

(iii) By Proposition 14.5, we have $E|X_n|^p \to E|X|^p$ and thus there exists $N_2 < \infty$ such that for all $n \geq N_2$,

$$E|X_n|^p - E|X|^p \leq \epsilon/3.$$

Choose $N_0 = \max\{N_1(M_1), N_2\}$. Then for any $n \geq N_0$, we have

$$\int_{\{|X_n|>M_1\}} |X_n|^p \, d\mathsf{P} \leq \epsilon.$$

Since N_0 is finite and $E|X_n|^p < \infty$ for all n, there exists $M_2 < \infty$ such that

$$\int_{\{|X_n| > M_2\}} |X_n|^p \, d\mathsf{P} \leq \epsilon, \qquad \text{for } n = 1, \ldots, N_0 - 1.$$

Hence, if we set $M_0 = (\max\{M_1, M_2\})^p$, we get

$$\sup_{n \geq 1} \int_{\{|X_n|^p > M_0\}} |X_n|^p \, d\mathsf{P} \leq \epsilon.$$

Since the choice of $\epsilon > 0$ is arbitrary, we conclude that the family $\{|X_n|^p\}$ is uniformly integrable.

The "if" part. The proof is similar to that for the "only if" part. Define a function φ_M by

$$\varphi_M(x) = \begin{cases} M & \text{if } x \geq M, \\ x & \text{if } |x| < M, \\ -M & \text{if } x \leq -M. \end{cases}$$

Some algebra and the triangle inequality yield

$$\begin{aligned} & \|X_n - X\|_p \\ \leq & \|\varphi_M(X_n) - \varphi_M(X)\|_p + \left(\int_{\{|X_n| > M\}} |X_n|^p d\mathsf{P} \right)^{1/p} + \left(\int_{\{|X| > M\}} |X|^p d\mathsf{P} \right)^{1/p}. \end{aligned}$$

Then bound the three terms on the right-hand side separately. To show $E|X|^p < \infty$ and then bound the last term, one can use the following version of Fatou's lemma: If $Y_n \geq 0$ and $Y_n \xrightarrow{P} Y$, then $\liminf_{n \to \infty} E(Y_n) \geq E(Y)$. $\square$

References

[1] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge university press, 2019.

[2] Sidney Resnick. *A Probability Path*. Springer, 2019.