Lecture 13

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For more details about the materials covered in this note, see Chapters 6.1, 6.2 and 8.5 of Resnick [1].

13.1 Convergence modes

Definition 13.1. Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and X be another random variable defined on the same space.

(i) We say X_n converges almost surely to X if $\mathsf{P}(\lim_{n\to\infty} X_n = X) = 1$, and we write $X_n \stackrel{a.s.}{\to} X$.

(ii) We say X_n converges in probability to X if $\lim_{n\to\infty} \mathsf{P}(|X_n - X| > \epsilon) = 0$ for any $\epsilon > 0$, and we write $X_n \stackrel{P}{\to} X$.

(iii) We say X_n converges in L^p to X if $\lim_{n\to\infty} E|X_n - X|^p = 0$, and we write $X_n \stackrel{L^p}{\to} X$.

Definition 13.2. Let X be a random variable with distribution function F, and $\{X_n\}_{n\geq 1}$ be a sequence of random variables where X_n has distribution function F_n. We say X_n converges in distribution (or converges weakly) to X if $\lim_{n\to\infty} F_n(x) = F(x)$, for every $x \in \mathbb{R}$ at which F is continuous. We write $X_n \stackrel{D}{\to} X$.

Remark 13.1. For convergence in distribution, random variables $X, X_1, X_2, \dots$ can be defined on different probability spaces.

Example 13.1. Let $Z_1, Z_2, \dots$ be i.i.d. Bernoulli random variables with $\mathsf{P}(Z_i = 0) = p$ and $\mathsf{P}(Z_i = 1) = 1 - p$, where $p \in (0, 1)$. Define $X_n = \max\{Z_1, \dots, Z_n\}$. Then $X_n \stackrel{a.s.}{\to} 1$. Note that for any $n < \infty$, $\mathsf{P}(X_n < 1) = p^n > 0$.

Example 13.2. Consider a sequence of random variables $X_1, X_2, \dots$ such that $\mathsf{P}(X_n = n) = 1/n$ and $\mathsf{P}(X_n = 0) = 1 - 1/n$. Then $X_n \stackrel{P}{\to} 0$ since, for any $\epsilon > 0$, $\mathsf{P}(|X_n| > \epsilon) = 1/n$ and converges to 0. But $E[X_n] = 1$ for every n. Hence, X_n does not converge to 0 in L^1. Indeed, X_n does not converge in L^p for any $p \geq 1$.

Example 13.3 (Continuation of Example 13.2)**.** Recall how we constructed such a sequence of random variables in Example 5.1: consider the probability space $([0,1], \mathcal{B}([0,1]), m)$ where m denotes the Lebesgue measure and define

$$X_n(\omega) = \begin{cases} n, & \text{if } \omega \in (0, 1/n), \\ 0, & \text{otherwise.} \end{cases}$$

Then for every ω, we have $\lim_{n\to\infty} X_n(\omega) = 0$, i.e. $X_n \stackrel{a.s.}{\to} 0$.

Example 13.4 (Continuation of Example 13.2)**.** However, if we assume $X_1, X_2, \dots$ are independent, then the sequence $\{X_n\}$ does not converge almost surely. We do not prove this claim here. Let's consider another construction which does not converge almost surely either. We still consider the probability space $([0,1], \mathcal{B}([0,1]), m)$. Let's construct $X_1, X_2, \dots$ as follows:

$$X_1 = \mathbb{1}_{(0,1)},\ X_2 = 2\mathbb{1}_{(0,1/2)},\ X_3 = 3\mathbb{1}_{(1/2,5/6)},\ X_4 = 4\left(\mathbb{1}_{(5/6,1)} + \mathbb{1}_{(0,1/12)}\right)\dots$$

See the figure on the next page, which plots $X_1, X_2, \dots, X_{50}$. So if $X_n(\omega)$ converges almost surely to zero, then there must exist some integer $N < \infty$ such that $\sum_{n=N}^{\infty} 1/n < 1$. But this is impossible.

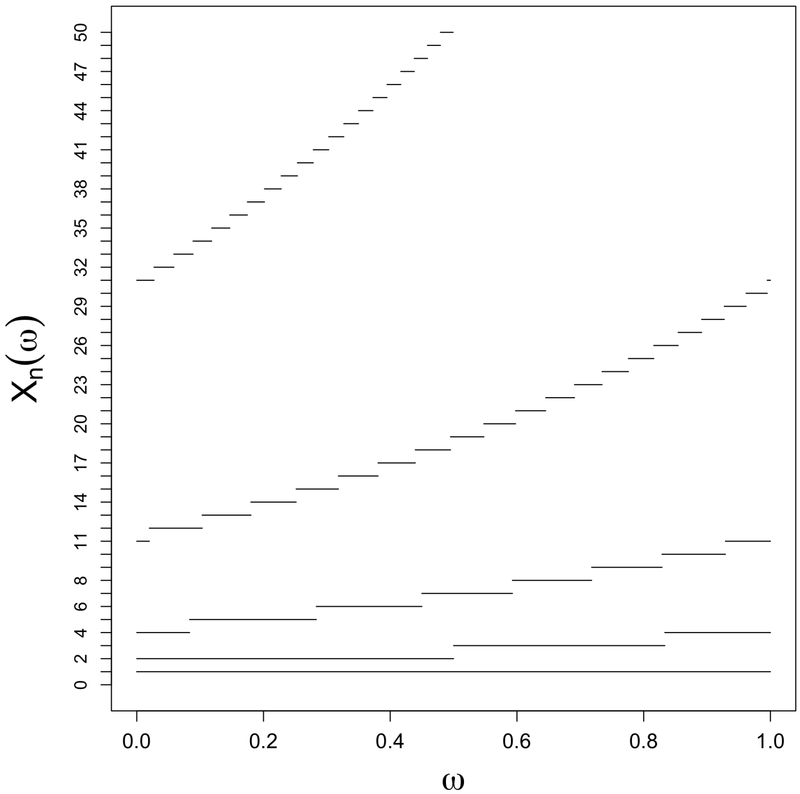
Example 13.5. Consider the probability space $([0,1], \mathcal{B}([0,1]), m)$ again. This time let's define a sequence of random variables by

$$X_1 = \mathbb{1}_{(0,1)}, \quad X_2 = \mathbb{1}_{(1/2,1)},$$
$$X_3 = \mathbb{1}_{(0,1/3)}, \quad X_4 = \mathbb{1}_{(1/3,2/3)}, \quad X_5 = \mathbb{1}_{(2/3,1)},$$
$$X_6 = \mathbb{1}_{(0,1/4)}, \quad X_7 = \mathbb{1}_{(1/4,1/2)}, \quad X_8 = \mathbb{1}_{(1/2,3/4)}, \quad X_9 = \mathbb{1}_{(3/4,1)}\dots$$

We do not plot these functions here, but one can check that X_n does not converge to 0 almost surely. However, for any $p \in (0, \infty)$, $E|X_n|^p = \mathsf{P}(X_n = 1) \to 0$; that is, $X_n \stackrel{L^p}{\to} 0$.

Example 13.6. Recall the Weak Law of Large Numbers (WLLN). If $X_1, X_2, \dots$ is an i.i.d. sequence of random variables with $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Then $\bar{X}_n \stackrel{P}{\to} \mu$.

Example 13.7. Let F be a distribution function for some continuous random variable. Define $F_n(x) = F(x+n)$ (check this is a distribution function!) Clearly, $F_n(x) \to 1$ for every $x \in \mathbb{R}$ since $\lim_{x\to\infty} F(x) = 1$. So $\{F_n\}$ is convergent but the limit is not a distribution function.



Example 13.8. Let X be a Bernoulli random variable with $\mathsf{P}(X = 0) = \mathsf{P}(X = 1) = 1/2$ and define a sequence of random variables $X_1, X_2, \dots$ by letting $X_n = X$. Clearly $X_n \xrightarrow{D} X$ (and it also converges in probability and almost surely.) Now let $Y = 1 - X$. Clearly, Y is another random variable with the same distribution as X and thus $X_n \xrightarrow{D} Y$. However, $\{X_n\}$ does not converge in probability to Y since $|X_n(\omega) - Y(\omega)| = 1$ for any n and ω.

13.2 Relations among modes of convergence

Theorem 13.1. *Let $\{X_n\}_{n \geq 1}, X$ be defined on the same probability space.*

(i) If $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{P} X$.

(ii) If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{D} X$.

(iii) If $X_n \xrightarrow{L^p} X$ for some $p > 0$, then $X_n \xrightarrow{P} X$.

(iv) If $X_n \xrightarrow{L^p} X$ for some $p > 0$, then $X_n \xrightarrow{L^r} X$ for $0 < r \le p$.

Proof of part (i). Almost sure convergence means that the event $\{\lim_{n\to\infty} X_n = X\}$ has probability 1, which implies that, for any $\epsilon > 0$, the event

$$B_\epsilon = \{|X_n - X| > \epsilon \text{ for infinitely many } X_n\}$$

has probability zero. Recall that given a sequence of sets $\{A_n\}$, we have that $\omega \in \limsup_{n\to\infty} A_n$ if and only if ω occurs in infinitely many A_n. Hence, we can write $B_\epsilon = \limsup_{n\to\infty}\{|X_n - X| > \epsilon\}$ and obtain

$$0 = \mathsf{P}(\limsup_{n\to\infty}\{|X_n - X| > \epsilon\}) = \mathsf{P}\left(\bigcap_{n\ge 1}\bigcup_{k\ge n}\{|X_k - X| > \epsilon\}\right).$$

By the continuity and monotonicity of measures,

$$\begin{aligned}\mathsf{P}\left(\bigcap_{n\ge 1}\bigcup_{k\ge n}\{|X_k - X| > \epsilon\}\right) &= \lim_{n\to\infty}\mathsf{P}\left(\bigcup_{k\ge n}\{|X_k - X| > \epsilon\}\right)\\ &\ge \limsup_{n\to\infty}\mathsf{P}(|X_n - X| > \epsilon).\end{aligned}$$

Hence, for any $\epsilon > 0$, we have $\limsup\limits_{n\to\infty}\mathsf{P}(|X_n - X| > \epsilon) = 0$. But this means $X_n \xrightarrow{P} X$. □

Proof of part (ii). Consider arbitrary $x \in \mathbb{R}$ and $\epsilon > 0$. If $X_n \le x$, then we have either $|X_n - X| > \epsilon$ or $X \le x + \epsilon$. Hence,

$$\{X_n \le x\} \subset \{X - x \le \epsilon\} \cup \{|X_n - X| > \epsilon\}.$$

This implies

$$\begin{aligned}F_n(x) = \mathsf{P}(X_n \le x) &\le \mathsf{P}(X - x \le \epsilon) + \mathsf{P}(|X_n - X| > \epsilon)\\ &= F(x+\epsilon) + \mathsf{P}(|X_n - X| > \epsilon).\end{aligned}$$

Similarly, $\{X \le x - \epsilon\} \subset \{X_n \le x\} \cup \{|X_n - X| > \epsilon\}$ and thus

$$F(x-\epsilon) \le F_n(x) + \mathsf{P}(|X_n - X| > \epsilon).$$

Letting n go to infinity and using the assumption that $X_n \xrightarrow{P} X$, we obtain

$$F(x-\epsilon) \le \liminf_{n\to\infty} F_n(x) \le \limsup_{n\to\infty} F_n(x) \le F(x+\epsilon).$$

If $F(x)$ is continuous at x, then $\lim_{\epsilon \to 0} F(x - \epsilon) = \lim_{\epsilon \to 0} F(x + \epsilon) = F(x)$, which yields

$$F(x) = \lim_{n \to \infty} F_n(x).$$

Hence, $X_n \xrightarrow{D} X$. □

Proof of part (iii). By Markov inequality,

$$\mathsf{P}(|X_n - X| \geq \epsilon) = \mathsf{P}(|X_n - X|^p \geq \epsilon^p) \leq \frac{E|X_n - X|^p}{\epsilon^p},$$

which converges to zero for any given $\epsilon > 0$. Hence, $X_n \xrightarrow{P} X$. □

Proof of part (iv). Recall that for $p \geq r$ and a random variable Z, we have $\|Z\|_{L^p} \geq \|Z\|_{L^r}$ (see Example 10.3). Hence,

$$E|X_n - X|^p \geq (E|X_n - X|^r)^{p/r} \geq 0.$$

Since $E|X_n - X|^p \to 0$, we have $E|X_n - X|^r \to 0$. □

References

[1] Sidney Resnick. *A Probability Path.* Springer, 2019.

[2] Jordan M Stoyanov. *Counterexamples in probability.* Courier Corporation, 2013.