Unit 7: Doob's Decomposition and Square Integrable Martingales

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7.1 Doob's decomposition

Theorem 7.1. *Let $(X_n)_{n\geq 0}$ be an adapted process with $\mathsf{E}|X_n| < \infty$ for every n. There exists an essentially unique decomposition $X_n = M_n + A_n$, where $(A_n)_{n\geq 0}$ is previsible with $A_0 = 0$ and $(M_n)_{n\geq 0}$ is a martingale. This is known as Doob's decomposition of (X_n). Further, (X_n) is a submartingale if and only if A is monotone non-decreasing a.s.*

Proof. The decomposition is given by

$$\begin{aligned} A_0 &= 0, \quad M_0 = X_0, \\ A_n &= \sum_{k=1}^{n} \left(\mathsf{E}[X_k \,|\, \mathcal{F}_{k-1}] - X_{k-1} \right), \quad \text{for } n \geq 1, \\ M_n &= X_0 + \sum_{k=1}^{n} \left(X_k - \mathsf{E}[X_k \,|\, \mathcal{F}_{k-1}] \right), \quad \text{for } n \geq 1. \end{aligned}$$

It is almost trivial to check that $X_n = M_n + A_n$ and (A_n) is previsible. To show (M_n) is a martingale, it suffices to notice that

$$\mathsf{E}\left[(X_n - \mathsf{E}[X_n \,|\, \mathcal{F}_{n-1}]) \,|\, \mathcal{F}_{n-1} \right] = 0.$$

To prove the uniqueness, suppose $X_n = \tilde{M}_n + \tilde{A}_n$ also satisfies the required conditions. We need to show that

$$\mathsf{P}(\tilde{M}_n = M_n, \, \tilde{A}_n = A_n \text{ for all } n) = 1.$$

(This is what we mean by "essentially unique".) Since $M_n + A_n = \tilde{M}_n + \tilde{A}_n$, $M_n - \tilde{M}_n$ is a previsible martingale. The uniqueness then follows from Exercise 3.1.

The definition of (A_n) clearly implies that (X_n) is a submartingale if and only if (A_n) is non-decreasing a.s. □

Example 7.1. Let $(X_n)_{n\geq 0}$ be the (symmetric) simple random walk; that is, $X_0 = 0$ and, for each $n \geq 1$, $X_n = X_{n-1} + Z_n$ where $Z_1, Z_2, \dots$ are i.i.d. such that $\mathsf{P}(Z_1 = 1) = \mathsf{P}(Z_1 = -1) = 1/2$. Since (X_n) is a martingale, $|X_n|$ is a submartingale. The Doob decomposition of $(|X_n|)$ is given by $|X_n| = M_n + A_n$ where

$$A_n = \sum_{k=1}^{n} (\mathsf{E}[|X_k| \,|\, \mathcal{F}_{k-1}] - |X_{k-1}|).$$

We can explicitly calculate that

$$\mathsf{E}[|X_k| \,|\, \mathcal{F}_{k-1}] - |X_{k-1}| = \begin{cases} 0, & \text{if } X_{k-1} \neq 0, \\ 1, & \text{if } X_{k-1} = 0. \end{cases}$$

Hence, A_n is simply the cardinality of the set $\{0 \leq k \leq n-1 \colon X_k = 0\}$, which is known as the local time of the process X at 0. This allows us to find that

$$\mathsf{E}|X_n| = \mathsf{E}A_n = \sum_{i=1}^{n-1} \mathsf{P}(X_i = 0) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{2j}{j} 4^{-j}.$$

Example 7.2. We can generalize the last example as follows. Let (X_n) be a stochastic process with initial value $X_0 = x_0$ such that $|X_n - X_{n-1}| = 1$ for all n. Let $f \colon \mathbb{Z} \to \mathbb{R}$ be a measurable function. Consider the process (Y_n) with $Y_n = f(X_n)$. Define the discrete derivatives of f by

$$f'(x) = \frac{f(x+1) - f(x-1)}{2}, \quad f''(x) = f(x-1) + f(x+1) - 2f(x).$$

One can check that the following holds, since $X_n = X_{n-1} \pm 1$:

$$f(X_n) - f(X_{n-1}) = f'(X_{n-1})(X_n - X_{n-1}) + \frac{1}{2} f''(X_{n-1}).$$

Letting $F'_n = f'(X_{n-1})$ and $F''_n = f''(X_{n-1})$, we can now write

$$f(X_n) = f(x_0) + (F' \cdot X)_n + \frac{1}{2} \sum_{i=1}^{n} F''_i.$$

This can be seen as the discrete version of Itô formula. Now suppose X is a martingale. Since both F' and F'' are previsible, we get the Doob decomposition $f(X_n) = M_n + A_n$, where $M_n = f(x_0) + (F' \cdot X)_n$ is the martingale and $A_n = \frac{1}{2} \sum_{i=1}^{n} F''_i$. To recover the result of Example 7.1, it suffices to note that, for $f(x) = |x|$, $f''(x) = 2$ if $x = 0$, and $f''(x) = 0$ otherwise.

7.2 Square integrable martingales

Definition 7.1. Let $(X_n)_{n\geq 0}$ be a square integrable martingale, where "square integrable" means that $\mathsf{E}X_n^2 < \infty$ for each n. Let $(A_n)_{n\geq 0}$ be the unique previsible process such that $(Y_n)_{n\geq 0}$ is a martingale, where $Y_n = X_n^2 - A_n$. We say (A_n) is the square variation process of X, and denote it by $\langle X \rangle_n = A_n$.

Remark 7.1. Since (X_n^2) is a submartingale whenever (X_n) is a martingale, $(\langle X \rangle_n)$ is also called the increasing process associated with X.

Theorem 7.2. *Let (X_n) be a square integrable martingale. Then,*

$$\mathsf{E}[\langle X \rangle_n] = \mathrm{Var}(X_n - X_0).$$

Proof. Try it yourself. □

Corollary 7.1. *Let (X_n) be a square integrable martingale with $X_0 = 0$. Let $\langle X \rangle_\infty = \lim_{n\uparrow\infty} \langle X \rangle_n$. Then, $\mathsf{E}[\sup_{n\geq 0} |X_n|^2] \leq 4\mathsf{E}\langle X \rangle_\infty$.*

Proof. Note $\langle X \rangle_\infty$ exists since $\langle X \rangle_n$ is non-decreasing. The result then follows from Theorems 6.1 and 7.2. □

Corollary 7.2. *Let (X_n) be a square integrable martingale. Then the following statements are equivalent:*

(i) $\sup_n \mathsf{E}X_n^2 < \infty$.

(ii) $\mathsf{E}[\langle X \rangle_\infty] < \infty$.

(iii) (X_n) *converges in* L^2.

(iv) (X_n) *converges almost surely and in* L^2.

Proof. Try it yourself. □

Theorem 7.3. *Let (X_n) be a square integrable martingale. On the event $\{\langle X \rangle_\infty < \infty\}$, almost surely, $X_\infty = \lim_n X_n$ exists and is finite.*

Proof. Pick $k > 0$, and define $T_k = \inf\{n \geq 0 \colon \langle X \rangle_{n+1} \geq k\}$, which is a stopping time since the square variation process is previsible. Consider the stopped process $Y_n^k = X_{n\wedge T_k}$. By Exercise 7.2, $\langle Y^k \rangle_n = \langle X \rangle_{n\wedge T_k} < k$, a.s. Since a stopped martingale is still a martingale, Corollary 7.2 shows that (Y_n^k) converges a.s. and in L^2. In particular, convergence in L^2 implies that the limit is a.s. finite. Hence, there exists a measurable set A with $\mathsf{P}(A) = 1$ such that (Y_n^k) converges as $n \to \infty$ for every k. But for every $\omega \in A \cap \{\langle X \rangle_\infty < \infty\}$, we can find sufficiently large k such that $T_k(\omega) = \infty$ and thus $X_n = Y_n^k$ for every n. Thus, X_n converges on the set $A \cap \{\langle X \rangle_\infty < \infty\}$. □

Remark 7.2. Recall the following theorem for the convergence of random series (a special case of Kolmogorov's three-series theorem): if $Z_1, Z_2, \dots$ are independent with $\mathsf{E}[Z_n] = 0$ for each n and $\sum_{n=1}^\infty \mathrm{Var}(Z_n) < \infty$, then $\sum_{n=1}^\infty Z_n$ converges a.s. It is easy to check that this result is just a special case of Theorem 7.3.

Theorem 7.4. *Let (X_n) be a square integrable martingale. On the event $\{\langle X \rangle_\infty = \infty\}$, almost surely, $X_n / \langle X \rangle_n$ converges to 0.*

Proof. Since $\langle X \rangle_n \geq 0$, the process $H_n = (1 + \langle X \rangle_n)^{-1}$ is previsible and bounded by 1. Define a martingale (W_n) by

$$W_n = \sum_{k=1}^n (X_k - X_{k-1}) H_k = (H \cdot X)_n.$$

Since

$$\begin{aligned}
\mathsf{E}[(W_n - W_{n-1})^2 \,|\, \mathcal{F}_{n-1}] &= H_n^2 (\langle X \rangle_n - \langle X \rangle_{n-1}) \\
&\leq \frac{\langle X \rangle_n - \langle X \rangle_{n-1}}{(1 + \langle X \rangle_{n-1})(1 + \langle X \rangle_n)} \\
&= \frac{1}{1 + \langle X \rangle_{n-1}} - \frac{1}{1 + \langle X \rangle_n},
\end{aligned}$$

we have $\langle W \rangle_\infty = \sum_{n=1}^\infty \mathsf{E}[(W_n - W_{n-1})^2 \,|\, \mathcal{F}_{n-1}] < \infty$, a.s. Hence, W_n converges a.s. to a finite limit. On the event $\{\langle X \rangle_\infty = \infty\}$, Kronecker's lemma yields that, $X_n H_n \to 0$, a.s., but this just means $X_n / \langle X \rangle_n \to 0$, a.s. □

Remark 7.3. It is not difficult to show that the strong law of large numbers for i.i.d. random variables in L^2 is just a special case of Theorem 7.4.

Exercise 7.1. Let (X_n) be a square integrable martingale. For any $m \leq n$,

$$\mathsf{E}[X_n^2 \,|\, \mathcal{F}_m] = X_m^2 + \mathsf{E}[(X_n - X_m)^2 \,|\, \mathcal{F}_m].$$

Exercise 7.2. Let (X_n) be a square integrable martingale and T be a stopping time. Define the process (Y_n) by $Y_n = X_{n \wedge T}$. Show that for every n, $\langle Y \rangle_n = \langle X \rangle_{n \wedge T}$, a.s.

Exercise 7.3. Let $Y_1, Y_2, \dots$ be independent random variables such that $\mathsf{E}Y_n = 1$ and $\mathsf{E}Y_n^2 < \infty$ for each n. Define $X_n = \prod_{i=1}^n Y_i$. Show that $(X_n)_{n \geq 1}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$ defined by $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and

$$\langle X \rangle_n = \sum_{i=1}^n X_{i-1}^2 \mathrm{Var}(Y_i), \text{ a.s.}$$

7.3 Levy's extension of the Borel-Cantelli lemma

Theorem 7.5. *Let $(B_n)_{n\geq 0}$ be a sequence of events such that $B_n \in \mathcal{F}_n$ for each n. Let $X_n = \sum_{i=1}^n \mathbb{1}_{B_i}$, and $X_\infty = \lim_{n\uparrow\infty} X_n$. Let $p_n = \mathsf{P}(B_n \,|\, \mathcal{F}_{n-1})$ and define $Y_\infty = \sum_{n=1}^\infty p_n$. Then, almost surely,*

(i) $Y_\infty < \infty$ implies $X_\infty < \infty$;

(ii) $Y_\infty = \infty$ implies $X_n / Y_n \to 1$.

Proof. Let $Y_n = \sum_{i=1}^n p_n$. Define a martingale (M_n) by $M_0 = 0$ and

$$M_n = X_n - Y_n = \sum_{i=1}^n \left(\mathbb{1}_{B_i} - \mathsf{P}(B_i \,|\, \mathcal{F}_{i-1}) \right), \quad \text{for } n \geq 1.$$

A direct calculation gives

$$\begin{aligned} \langle M \rangle_n &= \sum_{k=1}^n \left(\mathsf{E}[M_k^2 \,|\, \mathcal{F}_{k-1}] - M_{k-1}^2 \right) \\ &= \sum_{k=1}^n \mathsf{E}[(M_k - M_{k-1})^2 \,|\, \mathcal{F}_{k-1}] \\ &= \sum_{k=1}^n \mathsf{E}[(\mathbb{1}_{B_k} - p_k)^2 \,|\, \mathcal{F}_{k-1}] \\ &= \sum_{k=1}^n p_k (1 - p_k) \leq Y_n. \end{aligned}$$

Hence, $\langle M \rangle_\infty \leq Y_\infty$. Now consider three subcases.

(i) $Y_\infty < \infty$ and $\langle M \rangle_\infty < \infty$. By Theorem 7.3, M_n converges a.s. to a finite limit. Thus, $X_n = M_n + Y_n$ also converges a.s. to a finite limit.

(ii) $Y_\infty = \infty$ and $\langle M \rangle_\infty < \infty$. Since $X_n/Y_n = 1 + M_n/Y_n$, in this case we have $X_n/Y_n \to 1$, a.s.

(iii) $Y_\infty = \infty$ and $\langle M \rangle_\infty = \infty$. By Theorem 7.4, we have $M_n/\langle M \rangle_n \to 0$, a.s., which implies $M_n/Y_n \to 0$ and thus $X_n/Y_n \to 1$, a.s.

The proof is complete. □

Remark 7.4. We now show that the two Borel-Cantelli lemmas are special cases of Theorem 7.5. First, if $\sum_{n=1}^{\infty} \mathsf{P}(B_n) = \sum_{n=1}^{\infty} \mathsf{E}[p_n] < \infty$, we have $\sum_{n=1}^{\infty} p_n < \infty$, a.s.. Hence part (i) of Theorem 7.5 yields $X_\infty < \infty$, a.s. The second Borel-Cantelli lemma assumes independence among $B_1, B_2, \ldots$ and is clearly a special case of part (ii) of Theorem 7.5.

References

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