Unit 6: Convergence in L^p

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6.1 Doob's L^p inequality

Lemma 6.1. *Let $(X_n)_{n\geq 0}$ be a submartingale, and define $\bar{X}_n = \max_{0\leq i\leq n} X_i^+$. For any $c > 0$,*

$$c\,\mathsf{P}(\bar{X}_n \geq c) \leq \mathsf{E}[X_n \mathbb{1}_{\{\bar{X}_n \geq c\}}].$$

Proof. We fix n and let $T = n \wedge \inf\{k\colon X_k \geq c\}$, which is a bounded stopping time. By Theorem 4.3, $\mathsf{E}X_T \leq \mathsf{E}X_n$. Let $A = \{\bar{X}_n \geq c\}$, and observe that on the event A^c, we have $T = n$. Hence, $X_T - X_n = (X_T - X_n)\mathbb{1}_A$, and thus

$$\mathsf{E}[X_n \mathbb{1}_A] \geq \mathsf{E}[X_T \mathbb{1}_A] \geq c\,\mathsf{E}[\mathbb{1}_A],$$

which proves the asserted inequality. □

Theorem 6.1. *Let $(X_n)_{n\geq 0}$ be a submartingale and $p \in (1, \infty)$. Then,*

$$\mathsf{E}[\bar{X}_n^p] \leq \left(\frac{p}{p-1}\right)^p \mathsf{E}[(X_n^+)^p],$$

where $\bar{X}_n = \max_{0\leq i\leq n} X_i^+$.

Proof. We use truncation. Pick $M < \infty$ and define $Y_n = \bar{X}_n \wedge M$. Lemma 6.1 yields $\mathsf{P}(Y_n \geq y) \leq y^{-1}\mathsf{E}[X_n^+ \mathbb{1}_{\{Y_n \geq y\}}]$, since $\{Y_n \geq y\} = \{\bar{X}_n \geq y\}$ if $M \geq y$ and $\{Y_n \geq y\} = \emptyset$ if $M < y$. Hence,

$$\begin{aligned}
\mathsf{E}[Y_n^p] &= \int_0^\infty p y^{p-1} \mathsf{P}(Y_n \geq y)\mathrm{d}y \\
&\leq \int_0^\infty p y^{p-2} \mathsf{E}[X_n^+ \mathbb{1}_{\{Y_n \geq y\}}]\mathrm{d}y \\
&= \mathsf{E}\left[X_n^+ \int_0^\infty p y^{p-2} \mathbb{1}_{\{Y_n \geq y\}}\mathrm{d}y\right] \\
&= \frac{p}{p-1}\mathsf{E}\left[X_n^+ Y_n^{p-1}\right].
\end{aligned}$$

Hölder's inequality yields that

$$\mathsf{E}\left[X_n^+ Y_n^{p-1}\mathrm{d}y\right] \leq \left(\mathsf{E}[(X_n^+)^p]\right)^{1/p} \left(\mathsf{E}[(Y_n^{p-1})^{p/(p-1)}]\right)^{(p-1)/p}.$$

Combining the two inequalities above and using $\mathsf{E}[Y_n^p] < \infty$ due to truncation, we obtain that

$$(\mathsf{E}[Y_n^p])^{1/p} \leq \frac{p}{p-1} \left(\mathsf{E}[(X_n^+)^p]\right)^{1/p}.$$

To conclude the proof, let $M \to \infty$ and apply monotone convergence theorem. □

Corollary 6.1. *Let $(X_n)_{n\geq 0}$ be a martingale and $p \in (1,\infty)$. Then,*

$$\mathsf{E}\left[(\max_{0\leq i\leq n} |X_i|)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathsf{E}\left[|X_n|^p\right].$$

Proof. Apply Theorem 6.1 and Lemma 6.2 below. □

Remark 6.1. In Theorem 6.1 and Corollary 6.1, there is no assumption on the integrability of $|X_n|^p$. Indeed, Corollary 6.1 implies that, for a martingale (X_n) and $p \in (1,\infty)$, $\sup_n \mathsf{E}|X_n|^p < \infty$ if and only if $\mathsf{E}[\sup_n |X_n|^p] < \infty$. However, this no longer holds if $p = 1$.

Lemma 6.2. *Let (X_n) be a martingale and φ be a convex function such that $\mathsf{E}|\varphi(X_n)| < \infty$ for each n. Then, (Y_n) is a submatingale (w.r.t. the same filtration) where $Y_n = \varphi(X_n)$.*

Proof. Apply Jensen's inequality for conditional expectations. □

Remark 6.2. If (X_n) is only a submartingale, we need to require φ to be a non-decreasing convex function. Then, $(\varphi(X_n))$ is still a submartingale.

Exercise 6.1. Let $Z_1, Z_2, \dots$ be independent such that $\mathsf{E}Z_n = 0$ for every n. Define $S_n = Z_1 + \cdots + Z_n$, and $V_n = \mathrm{Var}(S_n) = \sum_{i=1}^n \mathsf{E}Z_i^2$. Prove Kolmogorov's inequality:

$$\mathsf{P}(\max_{1\leq i\leq n} |S_i| \geq c) \leq V_n/c^2.$$

Hint: use the submartingale (S_n^2).

Exercise 6.2. Consider the setting of Exercise 6.1. Assume that $|Z_n| \leq K$ for every n. Prove that

$$\mathsf{P}(\max_{1\leq i\leq n} |S_i| \leq c) \leq \frac{(c+K)^2}{V_n + (c+K)^2 - c^2} \leq \frac{(c+K)^2}{V_n}.$$

Hint: use Theorem 4.3 with the martingale $(S_n^2 - V_n)$ and stopping time $T = n \wedge \inf\{k\colon |S_k| > c\}$.

6.2 Convergence in L^p

Theorem 6.2. *Let X_n be a martingale with $\sup_n \mathsf{E}|X_n|^p < \infty$ for some $p > 1$. Then X_n converges almost surely and in L^p.*

Proof. The assumption implies that $\sup \mathsf{E}|X_n| < \infty$, and thus Theorem 5.1 shows that $X_n \overset{\text{a.s.}}{\to} X_\infty$. Define $X^* = \sup_{n \geq 0} |X_n|$, and observe that $|X_n - X_\infty|^p \leq (2X^*)^p$ by the triangle inequality. But by Corollary 6.1, $(X^*)^p$ is integrable. Hence, we can apply dominated convergence theorem to conclude that $\mathsf{E}[|X_n - X_\infty|^p] \to 0$. □

Example 6.1. Consider the branching process (X_n) defined in Example 5.1, and we still let $W_n = X_n/\mu^n$ and $W_\infty = \lim W_n$. Exercise 5.1 shows that, if $\mu \leq 1$, then $X_n = 0$ (and thus $W_n = 0$) for all sufficiently large n. Hence, $W_\infty = 0$ a.s.

Now consider the case $\mu > 1$, and assume $\mathrm{Var}(Z_{0,1}) = \sigma^2 \in (0, \infty)$. By Lemma 6.3 below, we have

$$\mathrm{Var}(X_n) = \mu^2 \mathrm{Var}(X_{n-1}) + \sigma^2 \mathsf{E}[X_{n-1}].$$

Hence, using $X_n = W_n \mu^n$, we get

$$\mathrm{Var}(W_n) = \mathrm{Var}(W_{n-1}) + \frac{\sigma^2}{\mu^{n+1}} \mathsf{E}[W_{n-1}] = \mathrm{Var}(W_{n-1}) + \frac{\sigma^2}{\mu^{n+1}}.$$

An induction argument shows that

$$\mathrm{Var}(W_n) = \sum_{i=1}^{n} \frac{\sigma^2}{\mu^{i+1}} = \frac{\sigma^2(1 - \mu^{-n})}{\mu(\mu - 1)} \leq \frac{\sigma^2}{\mu(\mu - 1)}.$$

which is finite for every n. Hence, (W_n) is a martingale bounded in L^2. Thus, W_n converges to W_∞ a.s. and in L^2, and $\mathsf{E}[W_\infty] = 1$.

Lemma 6.3. *Let $X_1, X_2, \dots,$ be i.i.d. and N be a non-negative integer-valued random variable independent of $(X_n)_{n \geq 1}$. Define $S_N = \sum_{i=1}^N X_i$. Suppose $\mathsf{E}X_1^2 < \infty$ and $\mathsf{E}N^2 < \infty$. Then, $\mathsf{E}S_N^2 < \infty$ and*

$$\mathrm{Var}(S_N) = \mathrm{Var}(N)(\mathsf{E}X_1)^2 + \mathsf{E}(N)\mathrm{Var}(X_1).$$

Proof. Try it yourself. □

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

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[3] David Williams. *Probability with martingales.* Cambridge university press, 1991.