Unit 4: Stopping Times and Stopped Processes

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4.1 Stopping times

Definition 4.1. Let $T \colon \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$ be measurable. We say T is a stopping time w.r.t. $(\mathcal{F}_n)_{n \geq 0}$ if $\{T \leq n\} \in \mathcal{F}_n$ for each $n < \infty$.

Lemma 4.1. *Let $T \colon \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$ be measurable. T is a stopping time w.r.t. $(\mathcal{F}_n)$ if and only if $\{T = n\} \in \mathcal{F}_n$ for each $n < \infty$.*

Proof. To prove the "only if" part, observe that $\{T = n\} = \{T \leq n\} \cap \{T \leq n-1\}^c$, which is in $\mathcal{F}_n$. For the "if" part, fix an arbitrary n, and we have

$$\{T \leq n\} = \bigcup_{k=0}^{n} \{T = k\} \in \mathcal{F}_n,$$

since $\{T = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$ for any $k \leq n$. □

Theorem 4.1. *Given a stopping time T, define*

$$\mathcal{F}_T = \{A \in \mathcal{F} \colon A \cap \{T \leq n\} \in \mathcal{F}_n, \ \textit{for any } n\}.$$

Then $\mathcal{F}_T$ is a σ-algebra. It is called the stopped σ-algebra (or the σ-algebra of T-past).

Proof. Try it yourself. □

Remark 4.1. Just like $\mathcal{F}_n$ contains all the information up to time n, $\mathcal{F}_T$ contains all the information up to time T (which is random). But the definition of $\mathcal{F}_T$ may look much more confusing. The following example may help explain why $\mathcal{F}_T$ is defined in this way.

Example 4.1. Let $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$ and T be a stopping time. Consider the event

$$A = \left\{ \max_{0 \leq k \leq T} X_k \geq 1 \right\}.$$

Then $A \in \mathcal{F}_T$, since, for any n,

$$A \cap \{T = n\} = \left\{ \max_{0 \leq k \leq n} X_k \geq 1 \right\} \cap \{T = n\} \in \mathcal{F}_n.$$

Exercise 4.1. Let $(X_n)_{n\geq 0}$ be given and define $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$ for each n. Let σ, τ be stopping times w.r.t. $(\mathcal{F}_n)$. Which of the following random variables are always stopping times w.r.t. $(\mathcal{F}_n)$?

(i) $T = 101$.

(ii) $T = \inf\{n \geq 0 \colon X_n \in [1, 2]\}$.[1]

(iii) $T = \sup\{n \leq 100 \colon X_n \geq 7\}$.

(iv) $T = \inf\{n \geq 0 \colon X_n \geq X_{n+5}\}$.

(v) $T = \sigma \wedge \tau$.

(vi) $T = \sigma + \tau$.

(vii) $T = \tau - 5$ (assuming $\tau \geq 5$, a.s.)

Exercise 4.2. Show that a stopping time T is $\mathcal{F}_T$-measurable.

4.2 Stopped processes

Definition 4.2. Given an adapted process $(X_n)_{n\geq 0}$ and a stopping time T, let $(X_{n\wedge T})_{n\geq 0}$ is called the stopped process. More explicitly, letting $Y_n = X_{n\wedge T}$, we have $Y_n(\omega) = X_{n\wedge T(\omega)}(\omega)$.

Remark 4.2. Here is another way to view $\mathcal{F}_T$. Let $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$ for each n and T be a stopping time. Then, $\mathcal{F}_T = \sigma((X_{n\wedge T})_{n\geq 0})$, i.e., the σ-algebra generated by the stopped process (proof is omitted).

Theorem 4.2. *If $(X_n)_{n\geq 0}$ is a supermartingale and T is a stopping time, then $(X_{n\wedge T})$ is also a supermartingale.*

Proof. Define $H_n = \mathbb{1}_{\{T\geq n\}}$ for $n \geq 1$. Since $\{T \geq n\} = \{T \leq n-1\}^c \in \mathcal{F}_{n-1}$, (H_n) is previsible. Further,

$$(H \cdot X)_n = \sum_{k=1}^{n} H_k(X_k - X_{k-1}) = \sum_{k=1}^{n\wedge T} X_k - X_{k-1} = X_{n\wedge T} - X_0.$$

By Theorem 3.1, $H \cdot X$ is a supermartingale, which implies $\mathsf{E}[X_{n\wedge T} \mid \mathcal{F}_{n-1}] \leq X_{(n-1)\wedge T}$ for each $n \geq 1$. □

[1]By convention, we set $\inf(\emptyset) = \infty$ and $\sup(\emptyset) = 0$.

Theorem 4.3. *Let (X_n) be a submartingale and T be a stopping time such that $\mathsf{P}(T \leq m) = 1$ for some $m < \infty$. Then,*

$$\mathsf{E}[X_0] \leq \mathsf{E}[X_T] \leq \mathsf{E}[X_m].$$

Proof. By Theorem 4.2, (Y_n) is a submartingale where $Y_n = X_{n \wedge T}$. Hence, $\mathsf{E}[Y_0] \leq \mathsf{E}[Y_m]$. But $Y_0 = X_0$ and $Y_m = X_T$, a.s. Hence, $\mathsf{E}[X_0] \leq \mathsf{E}[X_T]$.

Next, define $H_n = \mathbb{1}_{\{T<n\}}$, which is previsible, and thus $H \cdot X$ is a submartingale. It follows that $0 = \mathsf{E}[(H \cdot X)_0] \leq \mathsf{E}[(H \cdot X)_m]$. Since $(H \cdot X)_m = X_m - X_T$, we obtain the other direction of the asserted inequality. □

Corollary 4.1. *Let (X_n) be a martingale and T be a stopping time such that $\mathsf{P}(T \leq m) = 1$ for some $m < \infty$. Then, $\mathsf{E}[X_0] = \mathsf{E}[X_T]$.*

Proof. Use Theorem 4.3 and the fact that (X_n) is both a supermartingale and submartingale. □

Remark 4.3. Theorem 4.3 can be seen as a special case of the famous optional sampling theorem. We will prove in later lectures analogous results where the boundedness of T is replaced by weaker conditions.

References

[1] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

[2] Achim Klenke. *Probability theory: a comprehensive course.* Springer Science & Business Media, 2013.

[3] David Williams. *Probability with martingales.* Cambridge university press, 1991.