Unit 15: Itô Integral

Instructor: Quan Zhou

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A motivating example

Let B be a standard Brownian motion. What is $\int_0^1 B_t \, \mathrm{d}B_t$?

Let's discretize the time by choosing $0 = t_0 < t_1 < \cdots < t_n = 1$. Consider

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} B(t_{k-1})(B(t_k) - B(t_{k-1})) \\
&= \sum_{k=1}^{n} B(t_k)(B(t_k) - B(t_{k-1})) - \sum_{k=1}^{n} (B(t_k) - B(t_{k-1}))^2 \\
&= B_1^2 - S_n - \sum_{k=1}^{n} (B(t_k) - B(t_{k-1}))^2.
\end{aligned}
$$

A motivating example

So we obtain that

$$2S_n = B_1^2 - V_n, \text{ where } V_n = \sum_{k=1}^{n} (B(t_k) - B(t_{k-1}))^2.$$

Recall that $B(t_k) - B(t_{k-1}) \sim N(0, t_k - t_{k-1})$ and the increments are independent. So $\mathrm{E}[V_n] = 1$ for every n. Under mild conditions on $(t_k)_{1 \leq k \leq n}$, V_n or a subsequence of V_n converges a.s. to 1.

Hence, we find that $\int_0^1 B_t \, \mathrm{d}B_t = (B_1^2 - 1)/2$.

A motivating example

We approximated $\int_0^1 B_t \, \mathrm{d}B_t$ using

$$S_n = \sum_{k=1}^{n} B(t_{k-1})(B(t_k) - B(t_{k-1}))$$

The result would be different if we used

$$S_n = \sum_{k=1}^{n} B(t_k)(B(t_k) - B(t_{k-1})),$$

$$\text{or } S_n = \sum_{k=1}^{n} \frac{B(t_{k-1}) + B(t_k)}{2}(B(t_k) - B(t_{k-1})).$$

Itô integral

Let B be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P})$. We will construct integrals of the form

$$I_t(X) := \int_0^t X_s \, \mathrm{d}B_s,$$

for a class of X s.t. $(I_t(X))_{t \geq 0}$ is a continuous martingale.

We will assume that $\mathcal{F}_t$ is the completion of $\sigma((B_s)_{0 \leq s \leq t})$ (see [2, Chap. 2.7]). It still holds that $B_t - B_s$ is independent of $\mathcal{F}_s$ for $t > s$. Completeness is needed to show that, e.g., $(I_t(X))_{t \geq 0}$ is also adapted.

Two L^2 spaces

Definition 15.1

We say X is measurable, if $(\omega, t) \mapsto X_t(\omega)$ is a measurable mapping from $(\Omega \times [0,\infty), \mathcal{F} \times \mathcal{B}([0,\infty))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Now consider the space

$$\mathcal{L}^2(B) = \{X \colon X \text{ is measurable, adapted to } (\mathcal{F}_t)_{0 \le t < \infty} \text{ and } \|X\|_2 < \infty\},$$

where $\|X\|_2 = \|X\|_{2,\infty}$ and

$$\|X\|_{2,t}^2 = \mathsf{E}\left[\int_0^t X_s^2 \,\mathrm{d}s\right], \qquad \text{for } 0 \le t \le \infty.$$

$\|X\|_{2,t}$ is just the L^2-norm of X, if we treat X as a function defined on $(\Omega \times [0,t], \mathcal{F} \times \mathcal{B}([0,t]), \mathsf{P} \times \mathrm{Leb})$. If $t = \infty$, we replace $[0,t]$ with $[0,\infty)$.

Two L^2 spaces

For any fixed t (including $t = \infty$), $I_t(X)$ is a mapping from Ω to $\mathbb{R}$. We will show that for $X \in \mathcal{L}^2(B)$, $I_t(X)$ has $\|I_t(X)\|_2 < \infty$, where

$$\|I_t(X)\|_2^2 = \mathrm{E}[(I_t(X))^2].$$

In other words, $I_t(X)$ is an element in the L^2-space

$$\mathcal{L}^2(\Omega) = \left\{ Y \colon (\Omega, \mathcal{F}, \mathrm{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), \text{ and } \mathrm{E}Y^2 < \infty \right\}.$$

Construction of Itô integral: Part I

Assume that X can be expressed by

$$X_t(\omega) = \sum_{i=1}^{n} f_{i-1}(\omega) \mathbb{1}_{(t_{i-1}, t_i]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n$ and, for $i = 0, 1, \ldots, n-1$, $f_i : \Omega \to \mathbb{R}$ is bounded and $\mathcal{F}_{t_i}$-measurable. We say X is a simple or elementary function (or process), and use $\mathcal{E}$ to denote the space of all such functions.

Definition 15.2

For $X \in \mathcal{E}$, define

$$(I_t(X)) = \sum_{i=1}^{n} f_{i-1}(B_{t_i \wedge t} - B_{t_{i-1} \wedge t}).$$

Theorem 15.3

For $X \in \mathcal{E}$, $(I_t(X))_{0 \le t < \infty}$ is a martingale with continuous paths, and

$$\mathsf{E}[I_\infty(X)^2] = \mathsf{E}\left[\int_0^\infty X_s^2 \, \mathrm{d}s\right].$$

That is, $\|I_\infty(X)\|_2 = \|X\|_2$. Further, $I_\infty : \mathcal{E} \to \mathcal{L}^2(\Omega)$ is linear.

Proof.

Try it yourself. □

Construction of Itô integral: Part II

We now define $I_t(X)$ for $X \in \mathcal{L}^2(B)$ by approximating X using a sequence of functions $(X^n)_{n \geq 1}$ that converge in L^2, i.e.,

$$\lim_{n \to \infty} \|X^n - X\|_2 = 0.$$

We will build this approximation sequence "backwards". For each $X \in \mathcal{L}^2(B)$, we approximate it in L^2 using a sequence of functions X^n, say, in some class $\mathcal{A}_1$; Then, for each $X \in \mathcal{A}_1$, we approximate it in L^2 using a sequence of functions in some class $\mathcal{A}_2$. Triangle inequality implies that we can also approximate $X \in \mathcal{L}^2(B)$ in L^2 using functions in $\mathcal{A}_2$. Eventually, we will approximate X using simple functions.

Construction of Itô integral: Part II

Henceforth, we will use the notations $X(\omega, t)$ and $X_t(\omega)$ interchangeably.

Step 1: for each $X \in \mathcal{L}^2(B)$, define X^n by

$$X^n(\omega, s) = X(\omega, s)\mathbb{1}_{[0,n]}(s).$$

Then $X^n \in \mathcal{L}^2(B)$, X^n converges to X pointwise, and $\|X^n\|_2 \leq \|X\|_2$. So DCT implies that $\lim_{n\to\infty}\|X^n - X\|_2 = 0$.

Define $\mathcal{A}_1 = \{X \in \mathcal{L}^2(B) \colon \exists t < \infty \text{ s.t. } X_s = 0 \text{ for all } s \geq t\}$.

Construction of Itô integral: Part II

Step 2: for each $X \in \mathcal{A}_1$, define X^n by

$$X^n(\omega, s) = X(\omega, s) \wedge n.$$

Again, DCT yields that $\|X^n - X\|_2 \to 0$.

Define $\mathcal{A}_2 = \{X \in \mathcal{A}_1 \colon X \text{ is bounded}\}$.

Construction of Itô integral: Part II

Step 3: for each $X \in \mathcal{A}_2$, we approximate it using functions in

$$\mathcal{A}_3 = \{X \in \mathcal{A}_2 \colon s \mapsto X_s(\omega) \text{ is continuous for every } \omega\}.$$

We use "moving average" to smooth the function:

$$X^n(\omega, s) = n \int_{(s-n^{-1}) \vee 0}^{s} X(\omega, u) \mathrm{d}u.$$

By the fundamental theorem of calculus, for each ω, $X^n(\omega, s)$ converges to $X(\omega, s)$ for almost every s. Since X, X^n are bounded, $\|X^n - X\|_2 \to 0$.

A technical challenge is to analyze the measurability of X^n. Completeness of $\mathcal{F}_t$ is needed (see [2]).

Construction of Itô integral: Part II

Step 4: for each $X \in \mathcal{A}_3$, we approximate it using functions in $\mathcal{E}$. Define $X_0^n = X_0$, and for $0 < s \leq t$, define

$$X^n(\omega, s) = \sum_{k=0}^{2^n-1} X\left(\omega, \frac{kt}{2^n}\right) \mathbb{1}_{(kt/2^n,(k+1)t/2^n]}(s).$$

For each ω, since $X(\omega)$ is continuous, we have $\lim_{n\to\infty} X^n(\omega, s) = X(\omega, s)$ for each s. Since X^n, X are bounded, $\lim_{n\to\infty}\|X^n - X\|_2 = 0$.

Conclusion: for each $X \in \mathcal{L}^2(B)$, there exists $(X^n)_{n\geq 1}$ in $\mathcal{E}$ such that $\lim_{n\to\infty}\|X^n - X\|_2 = 0$.

Construction of Itô integral: Part II

Definition 15.4

For $X \in \mathcal{L}^2(B)$, choose $(X^n)_{n \geq 1}$ in $\mathcal{E}$ such that $\lim_{n\to\infty} \|X^n - X\|_2 = 0$. Define

$$I_\infty(X) = L^2\text{-}\lim I_\infty(X^n),$$

where L^2-lim means that $\mathrm{E}[(I_\infty(X^n) - I_\infty(X))^2] \to 0$. Define

$$I_t(X) = I_\infty(X^{(t)}), \text{ where } X_s^{(t)} = X_s \mathbb{1}_{\{s \leq t\}}.$$

Since $(X^n)_{n\geq 1}$ converges in L^2, $(X^n)_{n\geq 1}$ is Cauchy in $\mathcal{L}^2(B)$. By Theorem 15.3, $(I_\infty(X^n))_{n\geq 1}$ is also Cauchy in $\mathcal{L}^2(\Omega)$. The space $\mathcal{L}^2(\Omega)$ is complete, which implies that $I_\infty(X)$ exists. Theorem 15.3 also implies that $I_\infty(X)$ is unique.

Construction of Itô integral: Part II

Theorem 15.5

For $X \in \mathcal{L}^2(B)$, $(I_t(X))_{0 \le t < \infty}$ is a martingale which has an a.s. continuous modification adapted to $(\mathcal{F}_t)_{t \ge 0}$, and $\|I_\infty(X)\|_2 = \|X\|_2$. Further, $I_\infty \colon \mathcal{L}^2(B) \to \mathcal{L}^2(\Omega)$ is linear.

The property $\|I_\infty(X)\|_2 = \|X\|_2$ is known as "Itô isometry", since it shows that I_∞ is a distance-preserving map from $\mathcal{L}^2(B)$ to $\mathcal{L}^2(\Omega)$.

Construction of Itô integral: Part II

Proof.

We only prove the martingale property. Let (X^n) be the approximating sequence of simple functions, which must satisfy, a.s.,

$$\mathrm{E}[I_\infty(X^n) \,|\, \mathcal{F}_t] = I_t(X^n).$$

But $I_t(X^n) \xrightarrow{L^2} I_t(X)$, and $\mathrm{E}[I_\infty(X^n) \,|\, \mathcal{F}_t] \xrightarrow{L^2} \mathrm{E}[I_\infty(X) \,|\, \mathcal{F}_t]$ by Jensen's inequality. The two limits must coincide; that is, a.s.,

$$\mathrm{E}[I_\infty(X) \,|\, \mathcal{F}_t] = I_t(X).$$

The fact that I_∞ is a linear isometry is easy to prove. For the existence of a continuous modification, see [2]. □

Construction of Itô integral: Part III

Recall that for $X \in \mathcal{L}^2(B)$ and fixed t, we defined $I_t(X)$ to be $I_\infty(X^{(t)})$, where $X_s^{(t)} = X_s \mathbb{1}_{\{s \le t\}}$.

Suppose $X \notin \mathcal{L}^2(B)$, but $\|X\|_{2,t} = \mathsf{E} \int_0^t X_s^2 \,\mathrm{d}s < \infty$. Intuitively, we should still be able to define the Itô integral $I_t(X)$, since $X^{(t)} \in \mathcal{L}^2(B)$.

To further generalize this idea using stopping times, we first prove some results which show that Itô integrals are indeed determined locally by the local values of the integrand.

Construction of Itô integral: Part III

Given a stopping time T and a process X, define the truncated process $X^{(T)}$ by $X_s^{(T)} = X_s \mathbb{1}_{\{s \le T\}}$.

Theorem 15.6

Let $X \in \mathcal{L}^2(B)$ and T be a stopping time. Then, $I_T(X) = I_\infty(X^{(T)})$ a.s.

Here $(I_T(X))(\omega)$ is interpreted as the value of the function $t \mapsto (I_t(X))(\omega)$ evaluated at $t = T(\omega)$. So the equivalence between $I_T(X)$ and $I_\infty(X^{(T)})$ is not obvious.

Construction of Itô integral: Part III

Sketch of proof.

Let $\mathcal{T}_n$ be the collection of all stopping times that take values in $\{k/2^n \colon k = 0, 1, \dots\} \cup \{\infty\}$.

Step 1: Assume that $T \in \mathcal{T}_n$ for some n and $X \in \mathcal{E}$. Directly use the definition of Itô integral to show that the equality holds.

Step 2: Assume that $T \in \mathcal{T}_n$ for some n and $X \in \mathcal{L}^2(B)$. Choose $X^n \in \mathcal{E}$ s.t. $\|X^n - X\|_2 \to 0$. Show that $I_\infty(X^{n,(T)}) \xrightarrow{L^2} I_\infty(X^{(T)})$. Let t be a possible value of T, and we have $I_t(X^n) \xrightarrow{L^2} I_t(X)$. On the event $\{T = t\}$, $I_\infty(X^{n,(T)}) = I_t(X^n)$ by step 1, and argue that the limits must equal a.s. Since there are countably many such t's, $I_\infty(X^{(T)}) = I_T(X)$, a.s.

Step 3: Approximate any stopping time T using $T_n = 2^{-n}\lceil 2^n T \rceil$. Since $T_n \in \mathcal{T}_n$, $I_\infty(X^{(T_n)}) = I_{T_n}(X)$ a.s. by step 2. Argue that $X^{(T_n)} \xrightarrow{L^2} X^{(T)}$, and use the path continuity of $I_t(X)$ to conclude the proof.

Construction of Itô integral: Part III

Corollary 15.7

Let T be a stopping time and $X, Y \in \mathcal{L}^2(B)$.

(i) *On the event $\{T \geq t\}$, $I_t(X) = I_t(X^{(T)})$, a.s.*

(ii) *If $X_t = Y_t$ for all $t \leq T$, $I_T(X) = I_T(Y)$, a.s.*

This result enables us to uniquely define the Itô integral for a larger class of integrands using stopping times.

Construction of Itô integral: Part III

Suppose X is measurable, adapted and

$$\mathsf{P}\left(\int_0^t X_s^2 \, \mathrm{d}s < \infty\right) = 1, \quad \forall\, t \geq 0.$$

To define the Itô integral of X, we can choose a sequence of stopping times $(T_n)_{n \geq 1}$ such that $T_n \uparrow \infty$ a.s. and $X^{(T_n)} \in \mathcal{L}^2(B)$. For example,

$$T_n = n \wedge \inf\left\{t \colon \int_0^t X_s^2 \, \mathrm{d}s \geq n\right\}.$$

Then, we define $I_t(X) = I_t(X^{(T_n)})$ for $0 \leq t \leq T_n$.

Construction of Itô integral: Part III

Theorem 15.8

Suppose X is measurable, adapted and $\|X\|_{2,t}^2 = \mathrm{E}[\int_0^t X_s^2 \, \mathrm{d}s] < \infty$ for every t. Then $(I_t(X))_{t \geq 0}$ is a square integrable continuous martingale.

Proof.

For any $0 \leq s < t$, since $\|X\|_{2,t}^2 < \infty$, we can define $I_s(X) = I_s(X^{(t)})$ and $I_t(X) = I_t(X^{(t)})$. Since $X^{(t)} \in \mathcal{L}^2(B)$, $\mathrm{E}[I_t(X) \,|\, \mathcal{F}_s] = I_s(X)$ by Theorem 15.5. So $(I_t(X))_{t \geq 0}$ is a martingale. □

Itô's lemma

For any $t > 0$, we can show that (see Exercise 15.1)

$$\sum_{k=1}^{2^n} \left(B_{kt/2^n} - B_{(k-1)t/2^n} \right)^2 \stackrel{\text{a.s.}}{\to} t, \qquad \text{as } n \to \infty.$$

This is the key difference between Brownian paths and a differentiable function. If we write $\mathrm{d}B_t = \sqrt{\mathrm{d}t}$, for smooth f, we have

$$\begin{aligned} \mathrm{d}f(B_t) &= f'(B_t)\mathrm{d}B_t + \frac{1}{2} f''(B_t)(\mathrm{d}B_t)^2 \\ &= f'(B_t)\mathrm{d}B_t + \frac{1}{2} f''(B_t)\mathrm{d}t. \end{aligned}$$

This is known as Itô's lemma.

Theorem 15.9

Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Almost surely,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s, \quad \forall t \geq 0.$$

Since f' and $t \mapsto B_t$ are both continuous, for each ω and t, $f'(B_s(\omega))$ is bounded on $s \in [0, t]$, and thus $\int_0^t f'(B_s)\,\mathrm{d}B_s$ is defined.
The main idea of the proof is to approximate $f(B_{s+h}) - f(B_s)$ using second-order Taylor expansion.

Examples

To recover the result for the motivating example considered at the beginning of this unit, let $f(x) = x^2/2$. Then,

$$\frac{1}{2}B_t^2 = \int_0^t B_s \, \mathrm{d}B_s + \frac{t}{2}.$$

Let $f(x) = x^4$. Then,

$$\mathrm{d}B_t^4 = 4B_t^3 \mathrm{d}B_t + 6B_t^2 \mathrm{d}t.$$

The martingale property of Itô integral yields $6\mathsf{E}[\int_0^t B_s^2 \mathrm{d}s] = \mathsf{E}[B_t^4]$.

Itô's lemma

Theorem 15.10

Let $f(x,s)\colon \mathbb{R}\times[0,\infty)\to\mathbb{R}$ *be in* $C^{2,1}$. *Almost surely,*

$$f(B_t,t) = f(B_0,0) + \int_0^t \frac{\partial}{\partial s} f(B_s,s)\,\mathrm{d}s + \int_0^t \frac{\partial}{\partial x} f(B_s,s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(B_s,s)\,\mathrm{d}s, \quad \forall t \geq 0.$$

Examples

Let $f(x,s) = e^{x-s/2}$ and $S_t = f(B_t, t)$. Then,

$$\frac{\partial}{\partial s} f(x,s) = -\frac{1}{2} f(x,s), \quad \frac{\partial}{\partial x} f(x,s) = f(x,s), \quad \frac{\partial^2}{\partial x^2} f(x,s) = f(x,s).$$

Hence,

$$\mathrm{d}S_t = S_t \mathrm{d}B_t.$$

S_t is called a geometric Brownian motion.

Itô's martingale representation theorem

Theorem 15.11

Suppose $(M_t)_{0 \leq t < \infty}$ *is a square integrable martingale w.r.t.* $(\mathcal{F}_t)_{t \geq 0}$. *Then there exists a measurable and adapted process* X *such that for every* t, $M_t = \mathrm{E}[M_0] + \int_0^t X_s \, \mathrm{d}B_s$ *a.s.*

For more general martingale representation results, see [2].

Exercises

Exercise 15.1

Let $t_{n,k} = k/2^n$ for $k = 0, 1, \ldots, 2^n$. Define

$$\tilde{V}_n = \sum_{k=1}^{2^n} (B(t_{n,k}) - B(t_{n,k-1}))^2.$$

Use Borel-Cantelli lemma to show that $\tilde{V}_n \stackrel{\text{a.s.}}{\to} 1$.

Exercise 15.2

Prove Theorem 15.3.

Exercises

Exercise 15.3

Show that $I_\infty(X)$ given in Definition 15.4 is unique.

Exercise 15.4

Use the definition of Itô integral to prove that $\int_0^t s \, \mathrm{d}B_s = tB_t - \int_0^t B_s \, \mathrm{d}s$.

Exercise 15.5

Show that $B_t^3 - 3tB_t$ is a martingale.

References

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